Bulletin of the London Mathematical Society Advance Access published January 5, 2010

BOOK REVIEW

General theory of Lie groupoids and Lie algebroids

(London Mathematical Society Lecture Note Series 213)

By Kirill C. H. Mackenzie: xxxviii+501 pp., £50.00 (US$90.00) (LMS members' price £37.50 (US$67.50)), ISBN 0-521-49928-3 (Cambridge University Press, Cambridge, 2005).

doi:10.1112/blms/bdp115

1. *The idea of groupoids. Some examples*

The notion of groupoid was discovered by the German mathematician Heinrich Brandt [**2**] in the 1920s, who also gave it its name. Brandt was led to this concept by his studies in number theory; it is worth stressing that his motivation was entirely practical†: to give an algebraic foundation to particular composition laws encountered by him in the theory of quaternary quadratic forms. In hindsight it is possible to find groupoids present implicitly in many earlier works. Pertinent to the present review: making explicit the vague notion of a 'continuous group of local transformations' used by S. Lie leads to (Lie) groupoids no less than to Lie groups.

Therefore, what is a groupoid? A loose description is a 'group with many objects'. While a group often appears as a symmetry group of some object, a groupoid can arise as a collection of symmetries linking several objects, such as fibers of a fiber bundle.

More precisely, a *groupoid* $\underline{G}$ consists of two sets, G_1 and G_0, called the *arrows* and the *objects*, respectively, with maps $\alpha, \beta : G_1 \to G_0$ called *source* and *target* (Mackenzie uses the original Ehresmann notation). It is equipped with a *composition* $\mu : G_1 * G_1 \to G_1$ defined on the subset $G_1 * G_1 = \{(g,h) \in G_1 \times G_1 \mid \alpha(g) = \beta(h)\}$; an *inclusion of objects* map $e : G_0 \to G_1$; and an *inversion* map $i : G_1 \to G_1$. Instead of $\mu(g,h)$, $e(x)$ and $i(g)$ one simply writes gh, 1_x and g^{-1}. The following properties hold: $\alpha(gh) = \alpha(h)$, $\beta(gh) = \beta(g)$, $\alpha(g^{-1}) = \beta(g)$, $\beta(g^{-1}) = \alpha(g)$; $g(hf) = (gh)f$ whenever both sides are defined; $g^{-1}g = 1_{\alpha(g)}$ and $gg^{-1} = 1_{\beta(g)}$, $g1_{\alpha(g)} = g = 1_{\beta(g)}g$. A diagram $G_1 \rightrightarrows G_0$ is often used for denoting a groupoid, with the source and target maps in mind. (Such notation hints at a link with simplicial methods, which indeed exists.) If G_0 is a singleton, then the groupoid is a group. The set G_0 is also called the *base* of a groupoid.

Groupoids can be regarded (and defined) as categories in which all morphisms are invertible. However, they appeared some 20 years before categories. The above form of the definition is a remake; one can start, as Brandt did, from a partial composition law on a set G_1 and require associativity and existence of inverses—then the left and right *Einheiten*, gg^{-1} and $g^{-1}g$ for all $g \in G_1$ define the set of objects G_0.

Constructions of groupoids are abundant. Any set X can be viewed as a groupoid by setting $G_1 = G_0 = X$. At the opposite extreme, a groupoid can be obtained by setting $G_1 = X \times X$ and $G_0 = X$, with the composition law $(z,y)(y,x) = (z,x)$; it is called the *pair groupoid* of X. Any group action $G \times X \to X$ gives rise to a groupoid $G \times X \rightrightarrows X$ with the composition

† That Brandt was more of an 'old school' mathematician compared with his contemporaries, such as van der Waerden, is evident from his exchange with H. Weyl of 1933, now published, devoted to their joint effort to help Emmy Nöther, who was endangered by the new regime in Germany.

$(g, hx)(h, x) = (gh, x)$, known as the *action groupoid*. For a topological space X, continuous paths in X considered up to homotopy relative to the endpoints make a groupoid with respect to the composition of paths, $\Pi(X)$, called the *fundamental groupoid* of X, incorporating both the fundamental groups $\pi_1(X, x_0)$ and the universal covering space of X. It gives the most suitable language for homology with local coefficients and the van Kampen theorem, allowing, for the latter, far-reaching generalizations [**3**]. Paths in X considered just up to reparametrization give the *path groupoid* of X. We have to stop and refer the reader to the surveys of Brown [**3**] and Weinstein [**22**] for further examples and for an introduction to the general theory of groupoids; an extensive use of groupoids can be found in Connes's book [**6**].

2. *Lie groupoids and Lie algebroids*

A crucial fact is that groupoids can be defined in any category, as long as the 'subset of composable arrows' $G_1 * G_1 \subset G_1 \times G_1$ makes sense: in the general setting, it is a fibered product $G_1 \times_{G_0} G_1$, with respect to the diagram $\alpha, \beta : G_1 \rightrightarrows G_0$. Thus, a *Lie groupoid* is a groupoid in the category of smooth manifolds: both G_1 and G_0 should be manifolds, and all maps in the definition should be smooth, with the source and target maps surjective submersions, so that the pullback manifold $G_1 * G_1$ exists. All the examples above, formulated in the smooth category, give Lie groupoids. A remarkable hybrid of the pair groupoid $M \times M$ and the tangent bundle TM, called the 'tangent groupoid' of a smooth manifold M plays a prominent role in [**6**]. (Since its base is $M \times [0, 1]$, it does not exactly fit the above definition of a Lie groupoid.)

Lie groupoids, under the name 'differentiable groupoids', were first introduced in the 1950s by Ehresmann [**8**], who promoted them as a foundation of differential geometry. In particular, he promoted them as an alternative to principal bundles for connection theory. This brings forward the following important class of groupoids. Every groupoid $G_1 \rightrightarrows G_0$ defines a map $G_1 \to G_0 \times G_0$, combining the source and target maps, whose image is an equivalence relation[†] on the base space G_0. A groupoid is *transitive* if this map is an epimorphism in the respective category. For sets this simply means that every two objects in G_0 can be joined by an arrow in G_1. For example, the fundamental groupoid of a topological space is transitive if the space is path-connected. For Lie groupoids, transitivity implies that the map $\alpha^{-1}(x_0) \to G_0$, for any $x_0 \in G_0$, is a principal bundle, with group the set of all arrows g with $\alpha(g) = \beta(g) = x_0$. Conversely, any principal G-bundle $E \to M$ with a Lie group G defines a transitive Lie groupoid often called the *gauge groupoid* of E. This relation between transitive Lie groupoids and principal bundles is not a complete equivalence of the two notions, since in one direction the construction involves choices. There are more morphisms for transitive Lie groupoids than for principal bundles.

A *connection* on a transitive Lie groupoid $\underline{G}$ over M is just a morphism from the path groupoid of M to $\underline{G}$. A usual connection on a principal bundle is the same as a connection on its gauge groupoid.

Clearly, the power of Lie groupoids lies in the possibility of using infinitesimal methods. Objects playing the same role for Lie groupoids as Lie algebras play for Lie groups, called 'Lie algebroids', were first introduced by Jean Pradines in 1967. A *Lie algebroid* is a vector bundle $A \to M$ endowed with a Lie bracket on the space of sections satisfying the Leibniz rule $[u, fv] = a(u)f\,v + f\,[u, v]$, where a is a vector bundle map $A \to TM$, called the *anchor*. A purely algebraic counterpart under the name of *Lie pseudoalgebra* had appeared earlier in [**9**]. (It has been rediscovered many times and given many other names; see [**13**].) A Lie

---

[†] Therefore groupoids are also a generalization of equivalence relations, the viewpoint stressed by Grothendieck.

algebroid is *transitive* if the anchor is surjective. (This corresponds to the transitivity of Lie groupoids.) A precursor of Lie algebroids was a certain exact sequence of vector bundles with bracket structures introduced in the analytic context by Atiyah [**1**] in 1957. Pradines sketched an analogue of Lie theory for Lie groupoids and Lie algebroids in a series of short announcements [**17**–**19**].

Unlike Lie algebras, Lie algebroids are not always integrable, that is, arise from Lie groupoids (an integration to 'local' Lie groupoids is always possible). The first examples of this phenomenon were found by Almeida and Molino at the same time as Kirill Mackenzie discovered a cohomological obstruction to integrability. The full theory for the transitive case was given in [**12**]. An integration theory applicable to the general case was found only very recently by Crainic and Fernandes [**7**].

The reader should be warned that it would be a mistake to regard Lie groupoids and Lie algebroids as just a harmless extension of the notion of Lie groups and Lie algebras. The presence of a base space G_0 or M makes the departure much greater than it may seem. Two points can be made.

First, one should rather think of a *Lie group action* (and of the corresponding action groupoid) than of an isolated Lie group, as a model for generalization. In physics (physicists work, of course, with infinitesimal objects), Lie algebroids appear under the guise of symmetries of field theories that do not come from a Lie group or Lie algebra and do not present themselves separately from the space of fields.

Second, there is a flexibility lacked by Lie groups: the existence of non-trivial higher-dimensional objects, that is, double and multiple Lie groupoids and algebroids[†].

3. *The present book*

Like his 1987 book [**12**], the new book by Mackenzie is a milestone in Lie groupoid and Lie algebroid theory.

The earlier book [**12**] gave the first coherent exposition of the theory, providing proofs for statements only announced in the literature before that. It also contained many original results never published previously in journal form, including the integrability theory for the transitive case. At the same time that book concluded the first period of the development of Lie groupoids and Lie algebroids, when, in particular, the transitive groupoids and algebroids were regarded of prime importance.

A lot has changed in Lie groupoid theory between the publication of [**12**] and the present book. First and foremost, new areas of applications have been discovered, most notably in Poisson geometry. Transitive Lie groupoids have ceased to be considered the main class of Lie groupoids. New structures, for example, Poisson and symplectic groupoids, Lie bialgebroids, etc., have come to the fore.

The book under review is a unique, comprehensive exposition of the general theory of Lie groupoids and Lie algebroids as it now stands. Except for some parts treating general topological groupoids, omitted in the new book, it has completely superseded the earlier monograph [**12**]. As was the case for [**12**], the author has not hesitated to include many of his new results that have never appeared in journals[‡].

[†]A 'double group', that is, a group in the category of groups is just an Abelian group, as students learn from the first pages of topology textbooks. Passing from groups to groupoids changes this drastically.

[‡]All familiar with the 'research assessment' climate should appreciate such a selfless attitude of the author preferring, for the benefit of the mathematical community, to present the fruits of his work in book form instead of spreading them between a dozen isolated papers.

There does not exist a comparable text devoted to Lie groupoids and Lie algebroids. The books [4, 6] touch on certain aspects of them (although Connes does not use Lie algebroids). The book [16], in spite of the title, is mainly devoted to foliation theory and treats Lie groupoids only in the last two chapters, from the viewpoint of foliations.

The author has invested a lot of work in the structure and exposition; the result is a very thoughtfully written text, showing utmost care for both logic and motivation.

Let me say more about the structure. The book is divided into three parts of almost equal size. Each part treats its own aspect of Lie groupoid and algebroid theory.

Part 1 (Chapters 1–4) is devoted to the foundations of Lie groupoids and Lie algebroids. The author introduces definitions and examples of the main notions, including such concepts as transitivity and the components of a Lie groupoid, the bisections of Lie groupoids, the construction of a Lie algebroid $A(\underline{G})$ from a Lie groupoid $\underline{G}$, and the exponential map. The more 'algebraic' chapters 2 and 4 discuss constructions such as pull-backs, morphisms, products and quotients. Among new material included in the book is a discussion of morphisms between Lie algebroids over different bases (the notion is non-obvious; see [10]). This allows the author to establish the functoriality of the correspondence $\underline{G} \rightsquigarrow A(\underline{G})$ in full generality.

Building on the general theory of Part 1, the other two parts of the book are devoted to more specialized topics.

Part 2 (Chapters 5–8) treats the theory of transitive Lie groupoids and transitive Lie algebroids, except for Chapter 7 devoted to Lie algebroid cohomology. Material presented here should be of interest to many differential geometers. In Chapters 5 and 6 the author develops elegant frameworks for connection theory, alternative to standard approaches. He manages to separate neatly the infinitesimal and global aspects. Parallel transport is introduced as a construction on transitive Lie groupoids (Chapter 6); as already mentioned, transitive Lie groupoids and principal bundles are 'almost' equivalent notions. A neat infinitesimal connection theory is distilled and phrased entirely in terms of transitive Lie algebroids which are not *a priori* related to Lie groupoids or principal bundles. (This idea has its source in the 'Atiyah sequence' mentioned above.) Namely, a section of the surjective anchor map $A \to TM$ for such an algebroid $A \to M$ is regarded as a 'connection' in A; 'curvature' arises as the failure of this map to be a Lie algebroid morphism.[†] Connections in principal bundles and in vector bundles can be obtained as special cases. Chapter 7 develops for an arbitrary Lie algebroid analogues of the Cartan and Schouten calculi. In particular, a cohomology theory is constructed that is a simultaneous generalization of de Rham cohomology and Lie algebra cohomology. A beautiful application is the construction of a natural flat connection in the cohomology bundle of the Lie algebra bundle associated with a transitive Lie algebroid: the celebrated Gauss–Manin connection is a particular case of it. Part 2 is crowned by the exposition of Mackenzie's 'cohomological obstruction' in Chapter 8.

Part 3 (Chapters 9–12) is concerned with those topics in Lie groupoid and Lie algebroid theory that have grown out of its relation to Poisson geometry. This relation is twofold. Lie algebroids are equivalent to a special kind of Poisson manifolds; more precisely, the dual vector bundle A^* carries a linear Poisson structure, a generalization of the Lie–Poisson–Berezin–Kirillov bracket on $\mathfrak{g}^*$ for a Lie algebra $\mathfrak{g}$. On the other hand, for a Poisson manifold M, the bracket of 1-forms discovered in integrable systems in the 1980s makes the cotangent bundle T^*M a Lie algebroid. When it can be integrated to a Lie groupoid $G_1 \rightrightarrows M$, G_1 has a compatible symplectic structure. This was discovered, independently, by several authors: [11, 21, 23, 24], in the late 1980s. Such *symplectic groupoids* are of fundamental importance; they can be viewed as providing a 'group-like object' for non-linear Poisson brackets. (When one

[†] A far-reaching development of this idea has been used in the works of T. Strobl and A. Cattaneo–G. Felder on 'sigma-models' theory.

starts from the linear Poisson bracket on $\mathfrak{g}^*$, a corresponding symplectic groupoid is precisely T^*G for a Lie group G with the Lie algebra $\mathfrak{g}$.) Chapter 10 explains relations between Poisson structures and Lie algebroids. It is preceded by Chapter 9 developing the theory of *double vector bundles* — the notion appearing naturally in many constructions in geometry and applications (such as classical mechanics) and heavily used in the rest of the book. Chapter 11 is devoted to *Poisson groupoids*, which include Poisson–Lie groups and symplectic groupoids as special cases. The treatment here is entirely new: Mackenzie uses the Lie theory of Part 1 and the double vector bundle theory of Chapter 9 to give a streamlined and conceptual account of Poisson groupoid theory. Chapter 12 treats *Lie bialgebroids*, introduced by Mackenzie and Xu [**15**], which serve as infinitesimal objects for Poisson groupoids and are a non-trivial generalization of Drinfeld's Lie bialgebras. Part 3 directly brings the reader to modern research.

Each chapter of the book is accompanied by 'small print' containing very informative notes on the historical development of the notions discussed, with comments on alternative approaches and hints to related areas of research, all with detailed references. The bibliography at the end of the book contains more than 250 items.

As mentioned, the whole book (especially Part 3) contains many previously unpublished original results due to the author. Besides that, throughout the book the author has worked hard to find a new streamlined exposition for the results already known from journals. This has been a gigantic task. The outcome is without exaggeration a masterpiece; it will remain a definitive exposition of, and an encyclopedic reference for, the subject for years to come.

The reviewer is convinced that an acquaintance with the methods and applications of Lie groupoids and Lie algebroids is a must nowadays for any working differential geometer and mathematical physicist. Therefore Mackenzie's treatise has to receive its rightful place on their bookshelves.

The thought about physicists leads me to the only serious criticism that I can make about this book. The author obviously prefers the coordinate-free way of introducing main concepts. However, certain things (it seems to me) can be understood more simply if expressed in local coordinates. Definitely physicists would prefer to see more local coordinates and 'objects with indices' in this book in order to learn more easily from it.

In spite of its encyclopedic nature, the book under review cannot cover everything. Beyond its scope remain, for example, the 'higher-dimensional' theory of double and multiple Lie groupoids and the corresponding Lie algebroids pioneered by Mackenzie (see [**14**]); integration theory for general Lie algebroids [**7**]; Cattaneo–Felder work on sigma-models giving a constructive approach to Kontsevich's deformation quantization [**5**]; methods based on supermanifolds (see, for example, [**20**]); étale groupoids, Morita equivalences (treated in [**16**]), and relations with stacks and gerbes. Some of these, namely, general integration theory and the 'doubles' for Lie bialgebroids are touched on in the appendix to the book.

Readers of an older generation might remember that it used to be fashionable (starting with Godement's book on sheaves and algebraic topology) to postpone the exposition of some topics to the imaginary second volume that the author had no intention of writing. In this particular case may I express a dream that, breaking the rule, the author of this book present us with another volume in continuation of the present one, treating some of the above in that skillful and beautiful manner which I can but admire.

References

1. M. F. Atiyah, 'Complex analytic connections in fibre bundles', *Trans. Amer. Math. Soc.* 85 (1957) 181–207.

2. H. Brandt, 'Über eine Verallgemeinerung des Gruppenbegriffes', *Math. Ann.* 96 (1926) 360–366.

3. R. Brown, 'From groups to groupoids: a brief survey', *Bull. London Math. Soc.* 19 (1987) 113–134. Extended version at http://www.bangor.ac.uk/r.brown/groupoidsurvey.pdf.

4. A. Cannas da Silva and A. Weinstein, *Geometric models for noncommutative algebras* (American Mathematical Society, Providence, RI, 1999).

5. A. S. Cattaneo and G. Felder, 'A path integral approach to the Kontsevich quantization formula', *Comm. Math. Phys.* 212 (2000) 591–611.
6. A. Connes, *Noncommutative geometry* (Academic Press, San Diego, CA, 1994).
7. M. Crainic and R. L. Fernandes, 'Integrability of Lie brackets', *Ann. of Math.* (2) 157 (2003) 575–620.
8. Ch. Ehresmann, 'Œuvres complètes et commentées. I-1,2. Topologie algébrique et géométrie différentielle', *Cah. Topol. Géom. Différ. Catég* 24 (1983) xxix+601.
9. J.-C. Herz, 'Pseudo-algèbres de Lie. I', *C. R. Acad. Sci. Paris* 236 (1953) 1935–1937.
10. Ph. J. Higgins and K. Mackenzie, 'Algebraic constructions in the category of Lie algebroids', *J. Algebra* 129 (1990) 194–230.
11. M. V. Karasëv, 'Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets', *Izv. Akad. Nauk SSSR Ser. Mat.* 50 (1986) 508–538, 638.
12. K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry* (Cambridge University Press, Cambridge, 1987).
13. K. C. H. Mackenzie, 'Lie algebroids and Lie pseudoalgebras', *Bull. London Math. Soc.* 27 (1995) 97–147.
14. K. C. H. Mackenzie, 'Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids', Preprint, 2006, `arXiv:math.DG/0611799`, 34 pp.
15. K. C. H. Mackenzie and P. Xu, 'Lie bialgebroids and Poisson groupoids', *Duke Math. J.* 73 (1994) 415–452.
16. I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids* (Cambridge University Press, Cambridge, 2003).
17. J. Pradines, 'Théorie de Lie pour les groupoïdes différentiables. Calcul différenetiel dans la catégorie des groupoïdes infinitésimaux', *C. R. Acad. Sci. Paris Sér. A–B* 264 (1967) A245–A248.
18. J. Pradines, 'Géométrie différentielle au-dessus d'un groupoïde', *C. R. Acad. Sci. Paris Sér. A–B* 266 (1968) A1194–A1196.
19. J. Pradines, 'Troisième théorème de Lie les groupoïdes différentiables', *C. R. Acad. Sci. Paris Sér. A–B* 267 (1968) A21–A23.
20. Th. Voronov, 'Q-manifolds and Mackenzie theory: an overview', ESI Preprint 1952, 2007, arXiv:0709.4232 [math.DG].
21. A. Weinstein, 'Symplectic groupoids and Poisson manifolds', *Bull. Amer. Math. Soc.* (*N.S.*) 16 (1987) 101–104.
22. A. Weinstein, 'Groupoids: unifying internal and external symmetry. A tour through some examples', *Notices Amer. Math. Soc.* 43 (1996) 744–752.
23. S. Zakrzewski, 'Quantum and classical pseudogroups. I. Union pseudogroups and their quantization', *Comm. Math. Phys.* 134 (1990) 347–370.
24. S. Zakrzewski, 'Quantum and classical pseudogroups. II. Differential and symplectic pseudogroups', *Comm. Math. Phys.* 134 (1990) 371–395.

Theodore Voronov
School of Mathematics
University of Manchester
Oxford Road, Manchester
M13 9PL
United Kingdom

theodore.voronov@manchester.ac.uk