Affinoid structures and connections*

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Abstract

Affinoids, introduced by Weinstein in 1990, are equivalent to principal bundles with structure groupoid, to matched pairs of equivalence relations, and to those double groupoids which have been called principal by the author. This paper correspondingly calculates the infinitesimal structure of an affinoid in three ways: by imitating the Atiyah sequence construction for a standard principal bundle, as an instance of a matched pair of Lie algebroids, and as a special case of the author's notion of the double Lie algebroid of a double Lie groupoid (though this is done without reference to the general construction).

The notion of affinoid structure takes several different forms: as a ternary relation, and in the context of dual pairs in symplectic geometry, they were introduced by Weinstein [11]; as a generalization of principal bundles they were introduced by Kock [4] under the name of *pregroupoid*; as a form of Morita equivalence for groupoids they were introduced by Pradines [9] as *butterfly diagrams*. Forms of the notion, however, go back to the early part of the century; see the references in [11] and [4].

In [6, §3] we gave proofs of the equivalences between affinoid structures, butterfly diagrams and generalized principal bundles, using simple functorial constructions from groupoid and double groupoid theory. The key there was to regard an affinoid structure as a type of double groupoid, called *principal* in [6]. The interest of affinoid structures in the context of double groupoids is that the other two equivalent formulations, which are not overtly

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double structures, provide a means for testing the correctness of the Lie theory for double groupoids begun in [6]. In this paper we calculate the infinitesimal invariants associated with affinoid structures, butterfly diagrams and generalized principal bundles. It turns out that these are not equivalent. We show in §2 that the infinitesimal analogue of an affinoid structure, which we calculate by a procedure which is in fact a specialization of the construction of the double Lie algebroid of a double Lie groupoid [7], is equivalent to a pair of flat partial conjugate connections—as already indicated in [11, Remark 3.2]—and this seems to us the correct infinitesimal concept. The infinitesimal analogues of both butterfly diagrams and generalized principal bundles are significantly weaker.

Some of the results of §2 of this paper are special cases of results of Mokri [8]; this reflects the fact that an affinoid structure may be regarded as a vacant double Lie groupoid [6, §3]. The more direct proofs available in the present case are of independent interest, however. Affinoid structures with symplectic or Poisson structures have been studied by [12] and [1], amongst others.

Although this paper is closely related to the author's work on double Lie algebroids, it requires no knowledge of double groupoid theory, or of the notion of double Lie algebroid. We do make frequent use of the classes of morphisms for groupoids and Lie algebroids introduced by Pradines [10]: the form we use comes from [3].

After the introductory §1, the main results of the paper are in §2, on infinitesimal affinoid structures and infinitesimal butterfly diagrams. In §3 we briefly indicate that the notion of generalized Atiyah sequence does not seem likely to be of much interest.

1 Affinoid structures

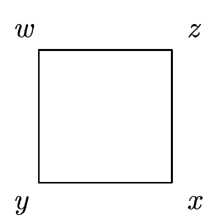
We recall the notion of affinoid structure from [11]; the description given here also makes use of the account in [6, §3].

Suppose given a set M and two surjections $h\colon M \to H$ and $v\colon M \to V$ to sets H and V. Call a pair $(x, y) \in M^2$ *vertical* if $h(x) = h(y)$; that is, if it belongs to the kernel pair

$$R(h) = \{(x, y) \in M^2 \mid h(x) = h(y)\}$$

of h. Similarly say that $(x, y) \in M^2$ is *horizontal* if $v(x) = v(y)$; that is, if it belongs to the kernel pair $R(v)$ of v.

Further suppose that we are given a subset $\Lambda \subseteq M^4$, whose elements are called *parallelograms*; display $(x, y, z, w) \in \Lambda$ as



Assume that if $(x, y, z, w) \in \Lambda$ then (x, z) and (y, w) are vertical and (y, x), (w, z) are horizontal. By way of converse, further assume that

Axiom I. *Given any $(x, y, z) \in M^3$ such that (y, x) is horizontal and (x, z) is vertical, there exists a unique $w \in M$ such that $(x, y, z, w) \in \Lambda$.*

Following [11], denote w by $yx^{-1}z$. Now define two elements (x, z), $(y, w) \in R(h)$ to be *v-parallel* if $(x, y, z, w) \in \Lambda$. Similarly, define (y, x), $(w, z) \in R(v)$ to be *h-parallel* if $(x, y, z, w) \in \Lambda$.

Axiom II. *The relations of v-parallelism and h-parallelism just defined are equivalence relations on $R(h)$ and $R(v)$ respectively.*

Definition 1.1 *An* affinoid structure *on a set M consists of two surjections $h\colon M \to H$ and $v\colon M \to V$, together with a subset $\Lambda \subseteq M^4$, which satisfy Axioms* I *and* II *above. We refer to H and V as the* bases *of the affinoid structure. A set equipped with an affinoid structure is an* affinoid space.

To define a differentiable affinoid space, first suppose that M, H and V are manifolds, and that h and v are surjective submersions. It follows that $R(h)$ and $R(v)$ are Lie subgroupoids of the pair groupoid M^2; in particular, the projection $R(h) \to M$, $(x, z) \mapsto x$, and the three others like it, are surjective submersions. We can therefore form the pullback $R(h) * R(v) \cong \{(x, y, z) \in M^3 \mid (x, z) \in R(h),\ (y, x) \in R(v)\}$.

Definition 1.2 *A* differentiable affinoid structure *on a manifold M is an affinoid structure on M such that h and v are surjective submersions, and Λ is a submanifold of M^4 such that the bijection $\Lambda \to R(h) * R(v)$ provided by Axiom* I *is a diffeomorphism.*

Henceforth we will only be concerned with differentiable affinoid structures and we will omit the adjective, unless emphasis is needed. Consider, then, an affinoid space M over H and V. As in [11], denote the set of equivalence classes of horizontal pairs modulo h-parallelism by G_h and the set of equivalence classes of vertical pairs modulo v-parallelism by G_v, and denote the projections by $\widetilde{h}\colon R(v) \to G_h$ and $\widetilde{v}\colon R(h) \to G_v$. Thus, modifying slightly the notation of [11], $\widetilde{h}(y, x) = yx^{-1}$ and $\widetilde{v}(z, x) = z^{-1}x$.

The kernel pair of $\widetilde{h}$ is precisely $\Lambda \subseteq R(h) \times R(h)$. Since Λ is a submanifold of $R(h) \times R(h)$ and the projection $\Lambda \to R(h)$, $(x, y, z, w) \mapsto (x, z)$, is a surjective submersion, it follows that G_h has a manifold structure making $\widetilde{h}$ a submersion. It is now easy to see that, with the structure defined in [11] or [6, §3], G_h is a Lie groupoid over H, with $\widetilde{h}$ a morphism over h. Similar remarks apply to the vertical structure.

There is a natural left action of G_h on $h\colon M \to H$ given by

$$(yx^{-1})z = yx^{-1}z,$$

where $\alpha(yx^{-1}) = h(x) = h(z)$. Similarly there is a right action of G_v on $v\colon M \to V$ given by

$$w(z^{-1}x) = wz^{-1}x,$$

where $\beta(z^{-1}x) = v(z) = v(w)$. These actions are free, and commute. The corresponding action morphisms are $\widetilde{h}$ and $\widetilde{v}$.

Now Λ may be given a groupoid structure with base M, called by Kock [4] the diagonal structure. For $(x, y, z, w) \in \Lambda$ the source is x and the target w. Given $(w, u, v, s) \in \Lambda$ the diagonal composition is

$$(w, u, v, s)(x, y, z, w) = (x, uw^{-1}y, zw^{-1}v, s).$$

When Λ is equipped with this structure we denote it by Λ_D.

There are now morphisms

$$\overline{h}\colon \Lambda_D \to G_h, \quad (x, y, z, w) \mapsto yx^{-1}, \qquad \overline{v}\colon \Lambda_D \to G_v, \quad (x, y, z, w) \mapsto z^{-1}x.$$

These are groupoid morphisms over h and v respectively and, further, are inductors—that is, the induced map into the groupoid pullback is a diffeomorphism [6, p.191]. Their kernels are respectively $R(h)$ and $R(v)$, embedded in Λ_D via the morphisms

$$1_v\colon R(v) \to \Lambda_D,\ (y, x) \mapsto (x, y, x, y), \qquad 1_h\colon R(h) \to \Lambda_D,\ (z, x) \mapsto (x, x, z, z).$$

All of this data is summarized in the butterfly diagram of Pradines shown in Figure 1.

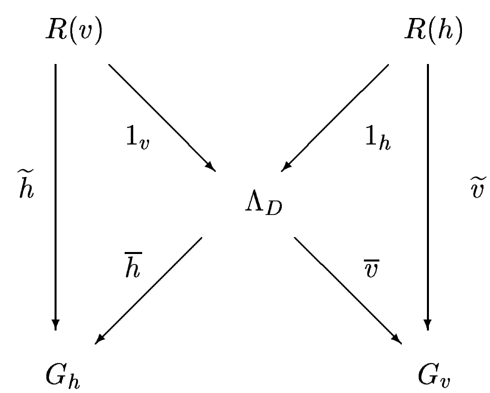


Figure 1:

A *butterfly diagram* [9] is a commutative diagram of groupoid morphisms of the form of Figure 1, such that the two vertical arrows are action morphisms, the two upper diagonals are closed embeddings over a fixed base, the two lower diagonals are inductors over surjective submersions, and the two sequences are exact at their central term. Any abstract butterfly diagram arises from an affinoid structure on the base of the central groupoid (see the end of [6, §3]). The concepts of affinoid structure and butterfly diagram are accordingly equivalent.

A third equivalent concept is that of generalized principal bundle: we deal with this in §3.

Example 1.3 Let $P(B,G,p)$ be a principal bundle and define

$$\Lambda = \{(x, yg, xg^{-1}, y) \mid x, y \in P, g \in G\}.$$

Then Λ is an affinoid structure on P with respect to $h = p\colon P \to B$ and $v\colon P \to \{*\}$ the constant map. Evidently $G_v \cong G$ and $G_h \cong (P \times P)/G$, the gauge groupoid of $P(B,G)$. Further, identifying (x, yg, xg^{-1}, y) with (y, g, x), the diagonal groupoid Λ_D is the trivial groupoid $P \times G \times P$.

There is a natural groupoid isomorphism $R(h) = P \times_p P \to P \times G$, $(x, xg) \mapsto (x, g)$. Here $P \times G$ is not the trivial group bundle but the action groupoid $P \rtimes G$ with structure $\alpha(x,g) = xg$, $\beta(x,g) = x$, $(y,h)(yh,g) = (y,hg)$.

2 Infinitesimal structures associated with affinoids

We first show, developing Remark 3.2 of [11], that associated with an affinoid structure on a manifold M, there is a distinguished submanifold $A^2 \subseteq T^2M$ which is a double vector bundle with side bundles T^hM and T^vM, and which may be regarded as a flat connection which is partial with respect to the two foliations induced by h and v. We abstract the properties of this structure into a concept of infinitesimal affinoid structure.

There is also a natural infinitesimal version of the concept of butterfly diagram. Provided that the fibres of h and v are simply–connected, we show that there is a bijective correspondence between infinitesimal affinoid structures and infinitesimal butterfly diagrams.

Consider an affinoid structure (Λ, H, V) on a manifold M. The morphism $\widetilde{v}\colon R(h) \to G_v$ defines a vertical subbundle $T^{\widetilde{v}}(R(h)) \subseteq T(R(h)) = R(T(h)) = TM \times_{TH} TM$. On the other hand, applying the Lie functor to the morphism $\widetilde{h}\colon R(v) \to G_h$ yields a Lie algebroid morphism $A(\widetilde{h})\colon T^vM \to AG_h$.

Proposition 2.1 $T^{\widetilde{v}}(R(h)) = R(A(\widetilde{h}))$.

PROOF. Take $(Z, X) \in T^{\widetilde{v}}(R(h))$. Thus $Z \in T_z(M)$, $X \in T_x(M)$ and

$$T(h)(Z) = T(h)(X), \qquad T(\widetilde{v})(Z, X) = 0.$$

Now $\widetilde{v}$ is a groupoid morphism over $v\colon M \to V$ and so, taking tangents,

$$\begin{array}{ccc} T(R(h)) & \xrightarrow{T(\widetilde{v})} & TG_v \\ {\scriptstyle T(\alpha')}\downarrow & & \downarrow{\scriptstyle T(\alpha)} \\ TM & \xrightarrow[T(v)]{} & TV \end{array}$$

commutes, where α' and α are the source projections. Now $T(\alpha')(Z, X) = X$ and so it follows that $T(v)(X) = 0$. Similarly, $Z \in T^vM$. It remains to prove that $A(\widetilde{h})(Z) = A(\widetilde{h})(X)$.

Regarding $\Lambda = R(\widetilde{v})$ as a groupoid on $R(h)$, it is easy to see that the maps $\Lambda \to R(v)$, $t\colon (x,y,z,w) \mapsto (w,z)$, and $s\colon (x,y,z,w) \mapsto (y,x)$, are morphisms. Applying the Lie functor, we have $A(s), A(t)\colon T^{\widetilde{v}}(R(h)) \to T^v M$ and $A(s)(Z,X) = X$, $A(t)(Z,X) = Z$. Now all that is necessary is to observe that on the groupoid level we have $\widetilde{h} \circ s = \widetilde{h} \circ t$.

This proves that $T^{\widetilde{v}}(R(h)) \subseteq R(A(\widetilde{h}))$. Since the maps $h, v, \widetilde{h}, \widetilde{v}$ are all submersions, a dimension count shows that equality holds. ■

Denote $T^{\widetilde{v}}(R(h))$ by $A_H\Lambda$. It is both a Lie algebroid on base $R(h)$, being an involutive distribution on $R(h)$, and a Lie groupoid on base $T^v M$, since it is the kernel pair of $A(\widetilde{h})$. In the terminology of [6, §4], $A_H\Lambda$ is an $\mathscr{LA}$–groupoid associated to the double groupoid structure on Λ.

Now define

$$A^2 = T^{A(\widetilde{h})} T^v M.$$

This is an involutive distribution on $T^v M$ but we prefer to regard it as a submanifold of $T^2 M$. Recall that this is a double vector bundle

$$\begin{array}{ccc} T^2M & \xrightarrow{T(p_M)} & TM \\ {\scriptstyle p_{TM}}\downarrow & & \downarrow{\scriptstyle p_M} \\ TM & \xrightarrow[p_M]{} & M, \end{array} \tag{1}$$

where the horizontal structure is obtained by applying the tangent functor to the operations in TM. We usually abbreviate p_M to p and p_{TM} to p_T. It is easy to see from the construction, or directly, that $T(p)$ maps A^2 to $T^h M$, and we consequently have a double vector bundle

$$\begin{array}{ccc} A^2 & \xrightarrow{T(p)} & T^hM \\ {\scriptstyle p_T}\downarrow & & \downarrow{\scriptstyle p} \\ T^vM & \xrightarrow[p]{} & M, \end{array} \tag{2}$$

which is a sub double vector bundle of (1).

Theorem 2.2 *The map $(T(p), p_T)\colon A^2 \to T^h M \oplus T^v M$ is a diffeomorphism.*

The proof of the following lemma is straightforward.

Lemma 2.3 *If $\varphi\colon \Omega' \to \Omega$ is a morphism of Lie groupoids over $f\colon M' \to M$, and $\varphi^!\colon \Omega' \to f^!\Omega$ is a diffeomorphism, then $A(\varphi)^!\colon A\Omega \to f^! A\Omega$ is also a diffeomorphism.*

Proof of 2.2: We know that $R(\widetilde{v}) \to R(h) * R(v)$, $(x,y,z,w) \mapsto ((z,x),(y,x))$, is a diffeomorphism, so $s\colon R(\widetilde{v}) \to R(v)$, $(x,y,z,w) \mapsto (y,x)$, which is a morphism over $R(h) \to M$, $(z,x) \mapsto x$, satisfies the condition of the lemma. Hence

$$T^{\widetilde{v}}(R(h)) \to R(h) * T^v M, \qquad (Z_z, X_x) \mapsto ((z,x), X),$$

is a diffeomorphism. Changing the point of view, it follows that the groupoid morphism $R(A(\widetilde{h})) \to R(h),\ (Z_z, X_x) \mapsto (z, x)$, over $T^vM \to M$ satisfies the condition of the lemma. Hence $A^2 \to T^vM \oplus T^hM$ is a diffeomorphism. ∎

Before proceeding, we recall some basic facts about connections [2, XVII§18]. Associated with the double vector bundle (1) are two exact sequences

$$p^!TM \rightarrowtail T^2M \xrightarrow{T(p)^!} \twoheadrightarrow p^!TM, \qquad p^!TM \rightarrowtail T^2M \xrightarrow{p_T^!} \twoheadrightarrow p^!TM$$

where the central terms are the vector bundles $p_T\colon T^2M \to TM$ and $T(p)\colon T^2M \to TM$ respectively. A *connection* in M is a map $\mathsf{C}\colon TM \oplus TM \to T^2M$ which is simultaneously a linear right–inverse for both sequences. We take it that $T(p)(\mathsf{C}(X,Y)) = Y$ and $p_T(\mathsf{C}(X,Y)) = X$. Given a connection C and a vector field X on M, define a vector field X^{C} on TM by $X^{\mathsf{C}}(Y) = \mathsf{C}(Y,X)$. The connection is *flat* if $[X,Y]^{\mathsf{C}} = [X^{\mathsf{C}}, Y^{\mathsf{C}}]$ for all $X, Y \in \mathcal{X}(M)$.

Given $X, Y \in \mathcal{X}(M)$ and $m \in M$, consider $T(Y)(X(m)) - X^{\mathsf{C}}(Y(m))$. This is a vertical tangent vector at $Y(m)$ and so corresponds to an element of $T_m(M)$, which is denoted $\nabla_X(Y)(m)$. This defines the associated *Koszul connection* ∇. There is a bijective correspondence between connections and Koszul connections.

Given a connection C, the *conjugate connection* C' is $\mathsf{C}' = J \circ \mathsf{C} \circ J_0$ where J is the canonical involution in T^2M and $J_0\colon TM \oplus TM \to TM \oplus TM$ interchanges the arguments. The corresponding Koszul connection ∇' is given by $\nabla'_X(Y) = \nabla_Y(X) + [X,Y]$.

We now consider the inverse of the diffeomorphism in Theorem 2.2 as constituting a "bipartial" connection in M adapted to the two foliations $R(h)$ and $R(v)$. More precisely, it is a partial connection in the vector bundle T^vM adapted to $R(h)$. The above observations about connections in M apply with the obvious modifications.

Let $\mathcal{X}^h$ and $\mathcal{X}^v$ denote the modules of sections of T^hM and T^vM. Given $X \in \mathcal{X}^h$ there is a unique $\overline{X} \in \Gamma_{T^vM}(A^2)$ which projects to X, and this induces as above an operator $\nabla^v_X\colon \mathcal{X}^v \to \mathcal{X}^v$. We thus obtain a "bipartial Koszul connection"

$$\nabla^v\colon \mathcal{X}^h \times \mathcal{X}^v \to \mathcal{X}^v.$$

Since $T(p)\colon A^2 \to T^hM$ is known to be a Lie algebroid morphism, we have $\overline{[X_1, X_2]} = [\overline{X_1}, \overline{X_2}]$ and so A^2 defines a flat bipartial connection. In terms of ∇^v we have $\nabla^v_{[X_1,X_2]} = [\nabla^v_{X_1}, \nabla^v_{X_2}]$ for all $X_1, X_2 \in \mathcal{X}^h$.

The construction of A^2 can be repeated with h and v interchanged and we obtain $A_2 = T^{A(\widetilde{v})}T^hM$ which is a double vector bundle

$$\begin{array}{ccc} A_2 & \xrightarrow{p_T} & T^hM \\ {\scriptstyle T(p)}\downarrow & & \downarrow{\scriptstyle p} \\ T^vM & \xrightarrow[p]{} & M, \end{array}$$

and a sub double vector bundle of (1). The proof of the following result is a straightforward check.

Proposition 2.4 *The canonical involution* $J\colon T^2M \to T^2M$ *carries* A^2 *isomorphically onto* A_2.

A_2 can be considered to be a flat partial connection in T^hM adapted to $R(v)$. In the same way as above we obtain a flat partial Koszul connection $\nabla^h\colon \mathcal{X}^v \times \mathcal{X}^h \to \mathcal{X}^h$.

The remainder of the section will be devoted to justifying the following definition.

Definition 2.5 *Let* M *be a manifold and let* $h\colon M \to H$ *and* $v\colon M \to V$ *be two surjective submersions. Then an* infinitesimal affinoid structure *on* (M, h, v) *is a sub double vector bundle* A^2 *of* T^2M *of the form* (2) *such that* $T(p)\colon A^2 \to T^hM$ *is a Lie algebroid morphism, and such that* $T(p)\colon A_2 = J(A^2) \to T^vM$ *is also a Lie algebroid morphism.*

Evidently any infinitesimal affinoid structure induces two partial Koszul connections ∇^v and ∇^h as above. By Proposition 2.4 and the definition of conjugate connections, we have

Proposition 2.6 *In any infinitesimal affinoid structure,* ∇^v *and* ∇^h *are conjugate connections; that is, for all* $X \in \mathcal{X}^h$, $Y \in \mathcal{X}^v$,

$$\nabla^h_Y(X) = \nabla^v_X(Y) + [X, Y].$$

Proposition 2.7 *For* $X, X_1, X_2 \in \mathcal{X}^h$, $Y, Y_1, Y_2 \in \mathcal{X}^v$,

$$\begin{aligned}
\nabla^v_X[Y_1, Y_2] &= [\nabla^v_X(Y_1), Y_2] + [Y_1, \nabla^v_X(Y_2)] + \nabla^v_{\nabla^h_{Y_2}(X)}(Y_1) - \nabla^v_{\nabla^h_{Y_1}(X)}(Y_2),\\
\nabla^h_Y[X_1, X_2] &= [\nabla^h_Y(X_1), X_2] + [X_1, \nabla^h_Y(X_2)] + \nabla^h_{\nabla^v_{X_2}(Y)}(X_1) - \nabla^h_{\nabla^v_{X_1}(Y)}(X_2).
\end{aligned}$$

PROOF. Since this is a purely formal calculation, we may as well consider an ordinary connection ∇ in M with conjugate connection ∇'. We calculate R', the curvature of ∇'.

By definition, $R'(X, Y)(Z) = \nabla'_{[X,Y]}(Z) - [\nabla'_X, \nabla'_Y](Z)$. Substituting into this the equation $\nabla'_X(Y) = \nabla_Y(X) + [X, Y]$, we obtain

$$\begin{aligned}
R'(X, Y)(Z) &= \nabla_Z[X, Y] - [X, \nabla_Z(Y)] - [\nabla_Z(X), Y]\\
&\quad -\nabla_{[Y,Z]}(X) - \nabla_{[Z,X]}(Y) - \nabla_{\nabla_Z(Y)}(X) + \nabla_{\nabla_Z(X)}(Y).
\end{aligned}$$

So if $R' = 0$ we get

$$\nabla_Z[X, Y] = [X, \nabla_Z(Y)] + [\nabla_Z(X), Y] + \nabla_{\nabla'_Y(Z)}(X) - \nabla_{\nabla'_X(Z)}(Y).$$

In the case of the bipartial flat connections of an infinitesimal affinoid we can apply this calculation to both ∇^v and ∇^h. ■

The equations of 2.7 show that T^hM and T^vM are a matched pair of Lie algebroids in the sense of Mokri [8]. The corresponding representations are ∇^h and ∇^v. Indeed (T^hM, T^vM) is the matched pair of Lie algebroids arising from the vacant double groupoid structure on Λ.

It is now clear that we could equivalently define an infinitesimal affinoid structure on (M, h, v) to be a pair of partial flat connections ∇^h and ∇^v which satisfy the equations of Proposition 2.7.

We now turn to the infinitesimal form of butterfly diagrams, which is easily defined.

Definition 2.8 *An* infinitesimal butterfly diagram *is a diagram of Lie algebroid morphisms of the form shown in Figure 2, such that the two vertical arrows are action mor-*

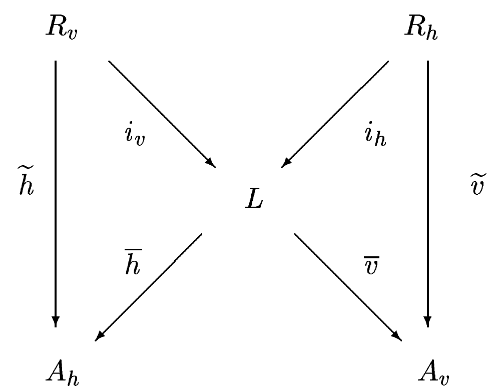


Figure 2:

phisms over surjective submersions $h\colon M \to H$ and $v\colon M \to V$, the two upper diagonals are base-preserving embeddings over h and v, the two lower diagonals are inductors, and the two sequences are exact at their central term.

Applying the Lie functor to a butterfly diagram clearly leads to an infinitesimal butterfly diagram, since the Lie functor sends action morphisms to action morphisms, inductors to inductors, embeddings to embeddings, and is exact [3].

We now show that an infinitesimal affinoid structure gives rise to an infinitesimal butterfly diagram, provided that the fibres of h and v are simply–connected. The first step holds without any connectivity assumption. Its proof is a straightforward calculation. (A more general result of this type is given in [8].)

Proposition 2.9 *Let (M, h, v) have an infinitesimal affinoid structure with associated partial connections ∇^h and ∇^v. Denote by L the vector bundle direct sum $T^hM \oplus T^vM$. Then L has a Lie algebroid structure over M with resepect to anchor $a\colon L \to TM$ given by $a(X \oplus Y) = X + Y$, and bracket*

$$[X \oplus Y, X' \oplus Y'] = \{[X, X'] + \nabla^h_Y(X') - \nabla^h_{Y'}(X)\} \oplus \{[Y, Y'] + \nabla^v_X(Y') - \nabla^v_{X'}(Y)\}$$

for $X, X' \in \mathcal{X}^h$, $Y, Y' \in \mathcal{X}^v$.

Until Theorem 2.10, consider an infinitesimal affinoid structure on (M, h, v) for which the fibres of h and v are simply connected. The partial connections ∇^h and ∇^v can be considered as Lie algebroid morphisms $T^vM \to \mathrm{CDO}(T^hM)$ and $T^hM \to \mathrm{CDO}(T^vM)$ and accordingly integrate to give linear actions of $R(v)$ on the vector bundle T^hM and of $R(h)$ on the vector bundle T^vM. Denote these actions by θ^h and θ^v.

We need to show that L quotients over T^vM and T^hM to give Lie algebroids A_v on V and A_h on H. We follow the method of [3, §4]. Say that $X \in \mathcal{X}^h$ is θ^h*-stable* if $\theta^h(y,x)X(x) = X(y)$ for all $(y,x) \in R(v)$. This is equivalent to the condition that X be ∇^h-parallel, that is, that $\nabla^h_Y(X) = 0$ for all $Y \in \mathcal{X}^v$. We also say that $X \oplus Y \in \Gamma L$ is θ^h-stable or ∇^h-parallel if X is so.

To show that T^vM is a v-ideal of L we need to verify the following three conditions:

(i) If $X \oplus Y,\ X' \oplus Y'$ are θ^h-stable, then $[X \oplus Y, X' \oplus Y']$ is also;

(ii) If $X \oplus Y \in \Gamma L$ is θ^h-stable, and $Y' \in \mathcal{X}^v$, then $[X \oplus Y, 0 \oplus Y']$ is in T^vM;

(iii) The map $L/T^vM \to TM/T^vM$ induced by the anchor of L is equivariant with respect to θ^h and the natural action of $R(v)$ on $TM/T^vM \cong v^!TV$.

The first two conditions are easily checked. For the third, note that the natural action of $R(v)$ on TM/T^vM differentiates to $D_Y(\overline{Z}) = \overline{[Y,Z]}$ where $Y \in \mathcal{X}^v,\ Z \in \mathcal{X}(M)$; here the bar denotes the class modulo T^vM. It therefore suffices to check that $\overline{\nabla^h_Y(X)} = \overline{[Y,X]}$ and this follows from the fact that $\nabla^h_Y(X) - [Y,X] = \nabla^v_X(Y) \in \mathcal{X}^v$.

From [3, 4.5] it therefore follows that the vector bundle $T^hM \cong L/T^vM$ descends to a vector bundle A_v on V; that is, T^hM is the vector bundle pullback of A_v over v. Denote the map $T^hM \to A_v$ by $\widetilde{v}$ and the map $L \to L/T^vM \to A_v$ by $\overline{v}$. Further, A_v has a Lie algebroid structure over V with respect to which $\widetilde{v}$ (and therefore $\overline{v}$) is a Lie algebroid morphism. The bracket of two sections $x, x' \in \Gamma A_v$ is obtained by taking the inverse image sections $X, X' \in \Gamma L$; by (i) above, $[X, X']$ is θ^h-stable and therefore descends to a section $[x, x']$ of A_v.

The rank of A_v is the same as that of T^hM and so $\overline{v}$ is an action morphism. Since $\widetilde{v}$ has kernel T^vM by construction, it is an inductor. Carrying out the same construction with T^hM as kernel, we have proved the following.

Theorem 2.10 *Let (M, h, v) have an infinitesimal affinoid structure and assume that h and v have simply connected fibres. Then the above construction yields an infinitesimal butterfly diagram.*

Conversely consider an infinitesimal butterfly diagram as in Figure 2. An inductor of Lie algebroids is essentially a pullback in the category of Lie algebroids, and so its kernel is the vertical bundle of the base map. Thus we have $R_v = T^vM$ and $R_h = T^hM$.

Since $\overline{h}\colon L \to A_h$ is an inductor over h, there is an isomorphism of Lie algebroids

$$L \to TM \times_{h^!TH} h^!A_h, \qquad Z \mapsto (a_L(Z), \overline{h}^!(Z)),$$

where a_L is the anchor of L. And since $\widetilde{h}\colon T^vM \to A_h$ is an action morphism, we know that $\widetilde{h}^!\colon T^vM \to h^!A_h$ is an isomorphism; denote its inverse by η. Now it is readily checked that

$$L \to T^h M \oplus T^v M, \qquad Z \mapsto (a_L(Z) - \eta(\overline{h}^!(Z))) \oplus \eta(\overline{h}^!(Z))$$

is an isomorphism of vector bundles, and that i_h and i_v are now represented by $X \mapsto X \oplus 0$ and $Y \mapsto 0 \oplus Y$. We can therefore apply the following lemma, whose proof is purely formal.

Lemma 2.11 *Let the vector bundle direct sum $T^h M \oplus T^v M$ have a Lie algebroid structure over M with respect to which $T^h M$ and $T^v M$ are Lie subalgebroids. Define $\nabla^h\colon \mathcal{X}^v \times \mathcal{X}^h \to \mathcal{X}^h$ and $\nabla^v\colon \mathcal{X}^h \times \mathcal{X}^v \to \mathcal{X}^v$ by*

$$[0 \oplus Y, X' \oplus 0] = \nabla^h_Y(X') \oplus -\nabla^v_{X'}(Y).$$

Then ∇^h and ∇^v are flat partial connections and satisfy the relations in Proposition 2.7.

This completes the proof of the following result.

Theorem 2.12 *Let L be an infinitesimal butterfly diagram over $h\colon M \to H$ and $v\colon M \to V$. Then the above construction yields an infinitesimal affinoid structure on M.*

Theorems 2.10 and 2.12 establish an equivalence between infinitesimal affinoid structures and infinitesimal butterfly diagrams when the fibres of h and v are simply connected.

Proposition 2.13 *Let L be an infinitesimal butterfly diagram over $h\colon M \to H$ and $v\colon M \to V$. Assume that A_h is integrable and that v has connected and simply–connected fibres. Then A_v is also integrable.*

PROOF. In general we have an isomorphism

$$L \cong TM \times_{h^! TH} h^! A_h = h^{!!} A_h.$$

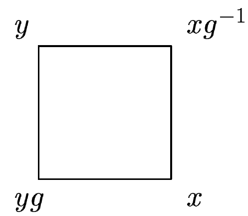
So if $A_h = A(G_h)$ then L is the Lie algebroid of the pullback groupoid $h^{!!} G_h$. Since $R(v)$ has connected and simply–connected fibres, $T^v M \to L$ integrates to an injective immersion $R(v) \to h^{!!} G_h$ and G_v may be constructed as the quotient of $h^{!!} G_h$ over $R(v)$. ■

3 Appendix: Generalized principal bundles and their Atiyah sequences

The following notion [9], [4] extends the concept of principal bundle to allow the structure group to be replaced by a structure groupoid.

Definition 3.1 *A* generalized principal bundle *consist of a Lie groupoid $G \rightrightarrows V$ acting smoothly and freely to the right on a surjective submersion $f\colon P \to V$ so that the quotient manifold $B = P/G$ exists. Denote the quotient map $P \to B$ by p.*

Given a generalized principal bundle $P(B,G,p)(V,f)$, define an affinoid structure Λ on P to consist of the parallelograms



where $x, y \in P$, $g \in G$, and $\beta g = f(y)$, $\alpha g = f(x)$. As in Example 1.3, this gives a differentiable affinoid structure with respect to $h = p\colon P \to B$ and $v = f\colon P \to V$. The vertical groupoid G_v identifies canonically with the original G by identifying each $y^{-1}(yg)$ and each $(xg^{-1})^{-1}x$ with g. For clarity, denote the elements of G_h by $\langle y, x\rangle$ where $x, y \in P$, $f(x) = f(y)$, and $\langle y, x\rangle = \langle yg, xg\rangle$ for any $g \in G$ with $\beta g = f(x)$.

Since the data of 3.1 is presented as a generalization of the concept of principal bundle, it is reasonable to extend the notion of Atiyah sequence to it. (The omitted details in what follows may be found by extending the account in [5, App.A].) Consider the vertical tangent bundle $T^f P$. The action of G on P lifts to a right action of G on $T^f P$ and remains free. Denote elements of $T^f P/G$ by $\langle X_x\rangle$ where $X_x \in T^f_x P$. Then $T^f P/G$ is a vector bundle over B; if $\langle X_x\rangle$ and $\langle Y_y\rangle$ have $p(x) = p(y)$, then there exists $g \in G$ with $y = xg$ and we define $\langle X_x\rangle + \langle Y_y\rangle = \langle Xg + Y\rangle$, as in the standard case. An f-vertical vector field X on P may be defined to be G*–invariant* if $X(xg) = X(x)g$ for all $x \in P$ and $g \in G$ with $f(x) = \beta g$; one obtains a $C(B)$–module of G–invariant vector fields which is in bijective correspondence with the module of sections of $T^f P/G \to B$. Now the bracket of G–invariant vector fields transfers to $\Gamma(T^f P/G)$ and makes it a Lie algebroid with anchor the quotient to $T^f P/G \to TB$ of $T(p)\colon T^f P \to TB$. This might be called the *generalized Atiyah sequence of* $P(B,G,p)(V,f)$. As in the standard case, $T^f P \to T^f P/G$ is an action morphism of Lie algebroids over p.

Theorem 3.2 *The generalized Atiyah sequence just constructed is canonically isomorphic to* $A(G_h)$.

PROOF. The morphism $\widetilde{h}\colon R(f) \to G_h$ over $p\colon P \to B$ is $(y, x) \mapsto \langle y, x\rangle$. It induces a Lie algebroid morphism $A(\widetilde{h})\colon T^f P \to A(G_h)$ which is constant on the orbits of G and therefore induces a Lie algebroid morphism $R\colon T^f P/G \to A(G_h)$ over B. Since $\widetilde{h}$ is an action morphism, it is a fibrewise diffeomorphism, and this property is inherited by $A(\widetilde{h})$ and R. Since R is base–preserving, it is therefore an isomorphism of Lie algebroids. ∎

Note that the method available in the standard case [5, III 3.20] cannot be used here, since P cannot generally be embedded in G_h.

Theorem 3.2 rules out much interest in the notion of generalized Atiyah sequence. Since any Lie groupoid G acts freely to the right on its target projection, yielding a generalized principal bundle whose G_h is again canonically isomorphic to G, any Lie algebroid which is the Lie algebroid of a Lie groupoid may be constructed as a generalized Atiyah sequence.

There is no prospect of extending to generalized Atiyah sequences the very rich theory known for the usual notion of Atiyah sequence, which depends essentially on transitivity.

Although generalized principal bundles are equivalent to both affinoid structures and butterfly diagrams (see [6, §3]), it is clear that generalized Atiyah sequences embody only a part of the information in an infinitesimal butterfly diagram.

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