2

Lie groupoids: Algebraic constructions

This chapter is concerned with quotients — in the most general sense, which includes descent constructions — with pullbacks, and with general semidirect products for Lie groupoids. Quotients and semidirect products in this sense are much more general, and of much wider importance, than the corresponding constructions for groups, in view of the possibility of changing the base manifold.

We begin in §2.1 by describing general quotients of vector bundles. This prefigures quotients of Lie algebroids as well as those of Lie groupoids. In §2.2 we briefly cover the case of base–preserving quotients of Lie groupoids; this very special case needs a separate treatment. In §2.3 we describe pullbacks of Lie groupoids; in principle the pullback construction allows most morphisms to be reduced to the base–preserving case.

The purpose of a notion of kernel is to characterize a class of morphisms (up to isomorphism) in terms of data entirely on their domain. The largest well behaved class of Lie groupoid morphisms for which this is possible is the fibrations; these are characterized by a lifting condition analogous to the classical notion of Hurewicz fibration. We prove in §2.4 that fibrations of Lie groupoids are characterized by what we call their kernel systems; for base–preserving fibrations, the kernel system is precisely the familiar kernel.

The notion of general semidirect product treated in §2.5 combines the action groupoid concept of Chapter 1 with the usual algebraic notion of semidirect product. This general notion of semidirect product corresponds to the notion of fibration: fibrations which are split in an appropriate sense correspond to general semidirect products.

§2.6 consists of a brief overview of the constructions of this chapter, and of the classes of morphism to which each corresponds.

2.1 Quotients of vector bundles

Given a vector bundle (E, q, M) and a smooth map $f\colon M' \to M$ we denote the inverse image bundle by $(f^!E, q^!, M')$ and the canonical projection $f^!E \to E$ by $f^!$. For $\mu \in \Gamma E$ we write $\mu^! \in \Gamma(f^!E)$ for the unique section such that $f^! \circ \mu^! = \mu \circ f$. The map

$$C^\infty(M') \otimes \Gamma E \to \Gamma(f^!E), \qquad u' \otimes \mu \mapsto u'\mu^!, \tag{1}$$

is an isomorphism of $C^\infty(M')$–modules, where the tensor product is over $C^\infty(M)$ and $C^\infty(M')$ is a $C^\infty(M)$–module under $uu' = (u \circ f)u'$.

The inverse image bundle $f^!E$ and the map $f^!$ have the universal property for vector bundle morphisms over f into E: namely, if $\varphi\colon E' \to E$ is any vector bundle morphism over f, then there is a unique vector bundle morphism $\varphi^!\colon E' \to f^!E$ over M' such that $\varphi = f^! \circ \varphi^!$. We thus often refer to $f^!E$ as the *pullback of E over f*. Any morphism $\psi\colon E' \to E$, $f\colon M' \to M$, which is a fibrewise bijection has this property and we usually identify such a E' and ψ with $f^!E$ and $f^!$ without comment. In particular, given any fibrewise bijection $\psi\colon E' \to E$ over $f\colon M' \to M$, and $\mu \in \Gamma E$, we may write $1 \otimes \mu$ or $\mu^!$ for the unique section of E' with $\psi \circ (1 \otimes \mu) = \mu \circ f$.

A fibrewise surjective morphism of vector bundles over a fixed base is determined (up to isomorphism) by its kernel. However this is no longer true for fibrewise surjections over general base maps.

Suppose $\varphi\colon E' \to E$ is a morphism of vector bundles over a surjective submersion $f\colon M' \to M$. If φ is fibrewise a surjection, then the union of the kernels $\cup_{m \in M} \ker(\varphi_m)$ will be precisely the kernel of the base–preserving morphism $\varphi^!\colon E' \to f^!E$. In particular, this union of the fibre kernels is a subbundle of E'.

This demonstrates that the kernel, defined as the union of the kernels of the maps of the fibres, cannot in general determine the image of a morphism. In order to describe a quotient appropriate to a change of base, extra data is needed.

In what follows we will restrict ourselves to morphisms the base maps of which are surjective submersions. This is the most general class of smooth maps which can usefully be regarded as quotient maps of manifolds. We recall the basic properties of quotient manifolds.

A surjective submersion $f\colon M' \to M$ determines an equivalence relation $R(f) = \{(n', m') \in M' \times M' \mid f(n') = f(m')\}$, sometimes called the *kernel pair* of f, which is a closed embedded regular submanifold of $M' \times M'$, and has the property that the projections $R(f) \to M'$ are surjective submersions. Conversely, given a manifold M', and an equivalence relation R whose graph is a closed embedded regular submanifold of $M' \times M'$ with projections $R \to M'$ which are submersions, there is a manifold $M = M'/R$ and a surjective submersion $f\colon M' \to M$ such that $R(f) = R$, and M and f are unique up to diffeomorphism with this property; this is the theorem of Godement.

Consider now a vector bundle morphism

$$\begin{array}{ccc} E' & \xrightarrow{\varphi} & E \\ q' \downarrow & & \downarrow q \\ M' & \xrightarrow[f]{} & M \end{array} \qquad (2)$$

and assume that f is a surjective submersion, and that φ is fibrewise surjective. Then $\varphi^!\colon E' \to f^!E$ is also fibrewise surjective, and so $E'/K \cong f^!E$, where $K = \ker(\varphi)$ is the kernel of $\varphi^!$. Denote the induced map $E'/K \to E$ by $\overline{\varphi}$, and the elements of E'/K by $\overline{e'}$, for $e' \in E'$.

The pullback bundle $f^!E$ is equipped with canonical identifications $f^!E|_{m'} \cong f^!E|_{n'}$ for all $(n', m') \in R(f)$; denote the map

$$f^!E|_{m'} \cong f^!E|_{n'}, \qquad (m', e) \mapsto (n', e), \qquad (3)$$

where $(n', m') \in R(f)$, by $\theta_0(n', m')$. It is clear that these maps constitute a linear action of the groupoid $R(f)$ on $f^!E$ in the sense of §1.7. For completeness, we repeat this case here.

Definition 2.1.1 Let (E', p', M') be a vector bundle and let $f\colon M' \to M$ be a surjective submersion. A *linear action of $R(f)$ on E'* is a collection of linear isomorphisms $\theta(n', m')\colon E'|_{m'} \to E'|_{n'}$, for $(n', m') \in R(f)$, such that:

(i) $\theta(m', m') = \mathrm{id}_{E'_{m'}}$ for all $m' \in M'$,

(ii) $\theta(m', n') = \theta(n', m')^{-1}$ for all $(n', m') \in R(f)$,

(iii) $\theta(p', n') \circ \theta(n', m') = \theta(p', m')$ for all $(p', n'), (n', m') \in R(f)$,

(iv) $R(f) \times_{M'} E' \to E'$, $((n', m'), a') \mapsto \theta(n', m')(a')$, is smooth, where the domain is the pullback manifold. □

Theorem 2.1.2 *Consider a vector bundle (E', q', M') and a surjective submersion $f\colon M' \to M$. Let θ be a linear action of $R(f)$ on E'. Then there exists a vector bundle (E, q, M), unique up to isomorphism over M, such that $E' \cong f^{!}E$ as vector bundles over M' with respect to an isomorphism which carries θ to the canonical action* (3) *of $R(f)$ on $f^{!}E$.*

Proof Define an equivalence relation $\sim$ on E' by

$$a' \sim b' \qquad \Longleftrightarrow \qquad (q'b', q'a') \in R(f) \text{ and } b' = \theta(q'b', q'a')(a').$$

Let $W \subseteq E' \times E'$ be the graph of $\sim$. We need to show that W is a closed embedded submanifold of $E' \times E'$ and that $\mathrm{pr}_1\colon E' \times E' \to E'$, restricted to W, is a surjective submersion; from this it will follow that $E'/\sim$ is a quotient manifold. First consider $\mathrm{id} \times q'\colon M' \times E' \to M' \times M'$ and define

$$W' = (\mathrm{id} \times q')^{-1}(R(f)).$$

Since $R(f)$ is a closed embedded submanifold of $M' \times M'$, the same is true of W' in $M' \times E'$. Now W is the image of $W' \to E' \times E'$, $(m', a') \mapsto (\theta(m', q'a')(a'), a')$, and this map is an immersion because the composite

$$W' \to E' \times E' \xrightarrow{q' \times \mathrm{id}} M' \times E'$$

is so. Indeed it also follows that $q' \times \mathrm{id}\colon W \to W'$ is a diffeomorphism and that W is an embedded submanifold of $E' \times E'$.

We can also regard W' as the pullback

$$\begin{array}{ccc} W' & \xrightarrow{\quad\pi\quad} & E' \\ {\scriptstyle \mathrm{id} \times q'}\downarrow & & \downarrow{\scriptstyle q'} \\ R(f) & \xrightarrow[\quad\mathrm{pr}_1\quad]{} & M' \end{array}$$

where π is $(m', a') \mapsto a'$. Now $\mathrm{pr}_1\colon R(f) \to M'$ is a surjective submersion, so π is also, and therefore the required projection

$$W \overset{\cong}{\to} W' \overset{\pi}{\to} E'$$

is also a surjective submersion. Thus $E = E'/\sim$ has a manifold structure with respect to which the natural map $\natural\colon E' \to E$, $a' \mapsto \overline{a'}$ is a surjective submersion.

The remainder of the proof is straightforward: there is a map $q\colon E \to M$ with $q \circ \natural = f \circ q'$ and it is a surjective submersion. The restriction of $\natural$ to each $E'_{m'} \to E_{f(m')}$ is a bijection and so the fibres of q acquire vector space structures. Vector bundle charts for E may be constructed by taking a chart $\psi'\colon U' \times V' \to E'_{U'}$ for E' and a local section $\sigma\colon U = f(U') \to U'$ of f and defining $\psi\colon U \times V' \to E_U$ by

$$\psi(m, v') = \overline{\psi'(\sigma(m), v')}.$$

The isomorphism $f^! E \to E'$ is $(m', \overline{a'}) \mapsto \theta(m', q'a')(a')$ and clearly preserves the actions of $R(f)$. □

The bundle (E, q, M) is the *quotient of* (E', q', M') *over* θ (or *over* $R(f)$), and is denoted E'/θ or $E'/R(f)$. This construction is also known as descent; we say that (E, q, M) is obtained from (E', q', M') by *descent over* $f\colon M' \to M$.

Since $\natural\colon E' \to E$ is a fibrewise bijection, each section $\mu \in \Gamma E$ induces a unique section $1 \otimes \mu$ of E' such that $\varphi \circ (1 \otimes \mu) = \mu \circ f$.

Lemma 2.1.3 *A section* μ' *of* E' *is of the form* $1 \otimes \mu$ *for some* $\mu \in \Gamma E$ *if and only if*

$$\theta(n', m')(\mu'(m')) = \mu'(n')$$

for all $(n', m') \in R(f)$.

A section satisfying this condition is said to be *θ–stable*. Thus $\mu' \in \Gamma E'$ is $\natural$–projectable if and only if it is θ–stable.

Given a morphism $\varphi\colon A \to B$ of vector bundles over M and any smooth map $f\colon M' \to M$, there is an induced morphism $f^!(\varphi)\colon f^! A \to f^! B$, $(m', a) \mapsto (m', \varphi(a))$ of the pullback bundles over M'. We now characterize morphisms which arise in this way (assuming that f is a surjective submersion).

Proposition 2.1.4 *Let* A' *and* B' *be vector bundles over* M' *with linear* $R(f)$ *actions* θ *and* δ, *where* $f\colon M' \to M$ *is a surjective submersion. Let* A *and* B *be the corresponding vector bundles on* M. *Then a morphism* $\varphi'\colon A' \to B'$ *of vector bundles over* M' *quotients (or descends) to a morphism* $\varphi\colon A \to B$ *over* M *if and only if* φ' *is* $R(f)$–*equivariant in the sense that*

$$\varphi'(\theta(n', m')(a')) = \delta(n', m')(\varphi'(a'))$$

for all $(n', m') \in R(f)$ *and* $a' \in A'$ *with* $q'a' = m'$.

Proof Define an $R(f)$ action η on $\mathrm{Hom}(A', B')$ by

$$\eta(n', m')(\psi) = \delta(n', m') \circ \psi \circ \theta(m', n').$$

Then the corresponding bundle on M is $\mathrm{Hom}(A, B)$ and the condition on φ asserts that it is η-stable as a section of $\mathrm{Hom}(A', B')$. □

Now return to the vector bundle morphism (2): φ is fibrewise surjective and f is a surjective submersion. The kernel of $\varphi^!$ is denoted K. Define a linear action θ of $R(f)$ on E'/K by $\theta(n', m')(\overline{e'}) = \overline{g'}$, where g' is any element of $E'_{n'}$ with $\varphi(g') = \varphi(e')$.

We will regard not only K, but also $R(f)$ and θ, as constituting the kernel of (2). To distinguish this from the usual concept, we call $(K, R(f), \theta)$ the *kernel system* of (φ, f). In general we use the following terminology.

Definition 2.1.5 Let (E', q', M') be a vector bundle. A *subbundle system* $\mathscr{K} = (K, R(f), \theta)$ of (E', q', M') consists of a vector subbundle K of E', a surjective submersion $f\colon M' \to M$, and a linear action θ of $R(f)$ on E'/K. □

Given a vector bundle (E', q', M') and a subbundle system $\mathscr{K} = (K, R(f), \theta)$ we can now form the quotient E'/K in the usual way and then apply 2.1.2 to obtain $E = (E'/K)/\theta$, a vector bundle over M. We denote the composition of the two quotient maps $E' \to E'/K$ and $E'/K \to (E'/K)/\theta$ by $\natural\colon E' \to E'/\mathscr{K}$. It is not hard to see that $\natural$ is fibrewise surjective and defines the kernel system $\mathscr{K}$.

The next result justifies calling $E = E'/\mathscr{K}$ the *quotient of E' over the subbundle system $\mathscr{K}$*.

Proposition 2.1.6 *Let (E', q', M') be a vector bundle and let $\mathscr{K} = (K, R(f), \theta)$ be a subbundle system with quotient $E = E'/\mathscr{K}$ on base M. Let $\varphi\colon E' \to E''$ be any morphism of vector bundles over any smooth map $g\colon M' \to M''$ such that:*

(i) $\varphi(K) = M'' \times \{0\}$,

(ii) $(g \times g)(R(f)) \subseteq \Delta_{M''}$,

(iii) *if $\overline{\varphi}$ is the induced morphism $E'/K \to E''$, then*

$$\overline{\varphi}(\theta(n', m')(\overline{e'})) = \overline{\varphi}(\overline{e'})$$

for all $(n', m') \in R(f)$ and $\overline{e'} \in E'/K$.

Then there is a unique vector bundle morphism $\psi\colon E \to E''$, $h\colon M \to M''$ such that $\psi \circ \natural = \varphi$ and $h \circ f = g$.

A morphism (φ, g) which satisfies the three conditions of 2.1.6 is said to *annul* $\mathscr{K}$.

Proof In (ii), $\Delta_{M''}$ is the diagonal of $M'' \times M''$. Condition (ii) implies that there is a well–defined set map $h\colon M \to M''$ with $h \circ f = g$; since f is a surjective submersion, h is smooth. Now the fibres of $\overline{\natural}\colon E'/K \to E$ are precisely the orbits of θ and so (iii) implies that there is a well–defined set map $\psi\colon E \to E''$ with $\psi \circ \overline{\natural} = \overline{\varphi}$. Again, ψ is smooth and it is routine to check that it is a vector bundle morphism over h. The uniqueness is clear. □

Clearly these three annulment conditions are necessary for the existence of (ψ, h).

Example 2.1.7 Let $f\colon M' \to M$ be a surjective submersion. Then $T(f)$ is a fibrewise surjection $TM' \to TM$ and the kernel of $T(f)^!$ is the vertical bundle $T^f M'$ of f, defined by

$$T^f_{m'}M' = \{X' \in T_{m'}(M') \mid T(f)(X') = 0\}.$$

Now suppose that M' is the 3–sphere in $\mathbb{R}^4$, regarded as pairs (α, β) of complex numbers with $|\alpha|^2 + |\beta|^2 = 1$. We will show that there is a trivial rank–2 bundle on $\mathbb{S}^3$ which descends under the Hopf map to $T\mathbb{S}^2$.

Tangent vectors to $\mathbb{S}^3$ at a point (α, β) can be regarded as pairs (ξ, η) of complex numbers such that, in terms of the standard inner product on $\mathbb{R}^4$, $\langle(\alpha, \beta), (\xi, \eta)\rangle = 0$. In terms of $\mathbb{C}^2$ this inner product is given by

$$\langle(\alpha, \beta), (\alpha', \beta')\rangle = \mathscr{R}(\alpha\overline{\alpha'} + \beta\overline{\beta'}).$$

It follows that $T_{(\alpha,\beta)}(\mathbb{S}^3)$ is generated by $(\alpha i, -\beta i)$, $(-\beta, \alpha)$ and $(\beta i, \alpha i)$. These are linearly independent on all of $\mathbb{S}^3$, which shows that $T\mathbb{S}^3$ is trivializable.

Let $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ act on $\mathbb{S}^3$ by

$$(\alpha, \beta)e^{it} = (\alpha e^{it}, \beta e^{-it})$$

and let $f\colon \mathbb{S}^3 \to M$ be the resulting quotient manifold and quotient map; M can be identified with $\mathbb{S}^2$.

At any point (α, β) of $\mathbb{S}^3$, the vectors tangent to the $U(1)$ orbit are generated by $(\alpha i, -\beta i)$. If we write E for the subbundle generated by $(-\beta, \alpha)$ and $(\beta i, \alpha i)$, it therefore follows that

$$T\mathbb{S}^3/T^f\mathbb{S}^3 \cong E.$$

Hence, by the discussion at the start of the example, $T(f)\colon E \to TM$ is a fibrewise isomorphism, and thus $T\mathbb{S}^2$ is obtained from E by descent. The action of $R(f)$ on E is

$$\theta((\alpha,\beta)e^{it},(\alpha,\beta))(\xi,\eta) = (\xi e^{it},\eta e^{-it}),$$

for any $(\xi,\eta) \in E$.

The groupoid $R(f)$ also acts on E via the trivialization of $T\mathbb{S}^3$; this action of course causes E to descend to $\mathbb{S}^2 \times \mathbb{R}^2$. ⊠

Lastly, consider a general fibrewise surjection $\varphi\colon E' \to E$, $f\colon M' \to M$, with associated kernel system $\mathscr{K} = (K, R(f), \theta)$. A section μ' of E' is *φ–projectable* if there exists $\mu \in \Gamma E$ such that $\varphi \circ \mu' = \mu \circ f$. Define $\varphi' \in \Gamma E'$ to be *θ–stable* if $\overline{\mu'} \in \Gamma(E'/K)$ is θ–stable; that is, for all $(n',m') \in R(f)$, we have $\theta(n',m')(\overline{\mu'}(n')) = \overline{\mu'}(m')$. The next result is a simple development of 2.1.3.

Proposition 2.1.8 *A section of E' is φ–projectable if and only if it is θ–stable.*

2.2 Base–preserving quotients of groupoids

A morphism of groups may be factored into a surjective morphism, followed by an isomorphism, followed by an injective morphism. For base–preserving morphisms of groupoids, a similar decomposition holds. This is straightforward and often sufficient, so we present this case first, starting with the algebra.

Definition 2.2.1 Let G be a groupoid on M. A *normal subgroupoid* of G is a wide subgroupoid $N \subseteq \mathscr{I}G$ of the inner subgroupoid such that for any $\nu \in N$ and any $g \in G$ with $\alpha g = \alpha\nu = \beta\nu$, we have $g\nu g^{-1} \in N$. □

Thus a normal subgroupoid is a collection of normal subgroups of the inner subgroupoid, which is invariant under the inner automorphisms of G.

Definition 2.2.2 Let $F\colon G \to G'$ be a base–preserving morphism of groupoids over M. Then the *kernel* of F is the set $\{g \in G \mid F(g) = 1_x \text{ for some } x \in M\}$. □

Clearly the kernel of a base–preserving morphism is a normal subgroupoid. The following construction of quotient groupoids shows that every normal subgroupoid is the kernel of a base–preserving morphism.

Proposition 2.2.3 *Let N be a normal subgroupoid of a groupoid G on M. Define an equivalence relation, denoted $\sim$, on G by*

$$g \sim h \iff \exists \nu \in N \text{ such that } h\nu = g.$$

Denote the equivalence classes by $[g]$, $g \in G$, and the set of them by G/N.

Then the following defines the structure of a groupoid with base M on G/N : the source and target projections are $\overline{\alpha}([g]) = \alpha(g)$, $\overline{\beta}([g]) = \beta(g)$; the object inclusion map is $x \mapsto [1_x]$; and the product $[h][g]$, where $\alpha(h) = \beta(g)$, is defined as $[hg]$. The inverse of $[g]$ is $[g^{-1}]$.

The projection $\natural\colon g \mapsto [g]$, $G \to G/N$, is a groupoid morphism over M with kernel N.

The proof is a straightforward modification of the corresponding result for group quotients and is left to the reader.

Note the extreme cases: $G/1_M$ is isomorphic to G under $\natural$, and $G/\mathscr{I}G$ is isomorphic to the image of (β, α) in $M \times M$.

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Now suppose that Ω is a locally trivial Lie groupoid on M and N is a closed embedded normal Lie subgroupoid of Ω. A section–atlas $\{\sigma_i\colon U_i \to \Omega_m\}$ for Ω induces charts ψ_i for $\mathscr{I}\Omega$ by $\psi_i(x, g) = \sigma_i(x) g \sigma_i(x)^{-1}$ where $g \in \Omega_m^m$, and the normality condition on N ensures that these restrict to $U_i \times N_m \to N_{U_1}$ and so N is a sub LGB of $\mathscr{I}\Omega$.

Let Γ denote the graph of the equivalence relation on Ω defined by N; thus

$$\Gamma = \{(\eta, \eta\nu) \mid \eta \in \Omega, \nu \in N, \alpha\eta = \beta\nu\}.$$

In place of the division map of 1.2.6, consider the other division map,

$$\delta'\colon \Omega \times_\beta \Omega \to \Omega, \quad (\eta, \xi) \mapsto \eta^{-1}\xi.$$

By a similar argument to 1.3.3, δ' is a surjective submersion, and so $\Gamma = \delta'^{-1}(N)$ is a closed embedded submanifold of $\Omega \times_\beta \Omega$, and hence of $\Omega \times \Omega$. Now the projection $\Gamma \xrightarrow{\mathrm{pr}_2} \Omega$ is a surjective submersion, since composing it with $\Omega * N \to \Gamma$, $(\eta, \nu) \mapsto (\eta\nu, \eta)$, gives the projection $\Omega * N \to \Omega$ onto the first factor, and this latter map is a surjective submersion because in the pullback square defining $\Omega * N$, the projection $N \to M$ is a surjective submersion.

So, by Godement's criterion, the quotient manifold Ω/Γ, which is Ω/N, exists.

It remains to prove that Ω/N is a Lie groupoid. Since $\overline{\alpha} \circ \natural = \alpha$, and α is a surjective submersion, it follows that $\overline{\alpha}$ is also. To prove that composition in Ω/N is smooth, note first that

$$(\natural \times \natural)^{-1}(\Omega/N * \Omega/N) = \Omega * \Omega$$

and therefore the restriction of $\natural \times \natural$ to $\Omega * \Omega \to \Omega/N * \Omega/N$ is a surjective submersion. Since

$$\begin{array}{ccc} \Omega * \Omega & \longrightarrow & \Omega \\ {\scriptstyle \natural \times \natural} \downarrow & & \downarrow {\scriptstyle \natural} \\ \Omega/N * \Omega/N & \longrightarrow & \Omega/N \end{array}$$

commutes, it follows that the multiplication in Ω/N is smooth. Thus we have the following result.

Theorem 2.2.4 *Let Ω be a locally trivial Lie groupoid on M and let N be a closed embedded normal Lie subgroupoid. Then Ω/N, with the structure just defined, is a locally trivial Lie groupoid, and $\natural\colon \Omega \to \Omega/N$ is a surjective submersion and a morphism.*

Lastly we note the following results, which are often useful.

Proposition 2.2.5 *Let $F\colon \Omega \to \Omega'$ be a morphism of locally trivial Lie groupoids over a connected base M. Then:*

(i) *F is of constant rank;*

(ii) *if F_m^m is injective for some $m \in M$, then F is an injective immersion;*

(iii) *if F_m^m is surjective for some $m \in M$, then F is a surjective submersion.*

Proof Let $\{\sigma_i\colon U_i \to \Omega_m\}$ be a section–atlas for Ω and consider the section–atlas $\{\sigma'_i = \varphi \circ \sigma_i\colon U_i \to \Omega'_m\}$ for Ω'. With respect to the corresponding charts, F is locally $\mathrm{id} \times f \times \mathrm{id}\colon U_i \times G \times U_i \to U_i \times G' \times U_i$ where $f = F_m^m$. The results thus follow from the corresponding statements for Lie groups. ☐

Example 2.2.6 Note that for general Lie groupoids a morphism may be a submersion into a connected codomain, but not be surjective; let M be the interval $(1,2)$ (say) in $\mathbb{R}$, let G be the multiplicative group of positive reals, and let $F\colon M \times M \to G$ be $(y,x) \mapsto yx^{-1}$. ⊠

The next result is proved using the same methods as for 2.2.4.

Proposition 2.2.7 *Let $\varphi\colon \Omega \to \Omega'$ be a morphism of locally trivial Lie groupoids over M. Then $K = \ker\varphi$ is a closed embedded submanifold of $\mathscr{I}\Omega$, and a Lie group subbundle of $\mathscr{I}\Omega$. Further,* $\mathrm{im}(\varphi)$ *is a submanifold of Ω' and a reduction of Ω'.*

For general Lie groupoids, not necessarily locally trivial, it would be useful to have a theory of quotients such that, for example, the quotient over the inner subgroupoid would be the image of the anchor with an appropriate structure. We do not attempt such a theory here.

2.3 Pullback groupoids

The general concept of morphism of groupoids was defined in §1.2. As a preliminary to the general quotients and semidirect products which follow, we express this in terms of pullbacks.

Consider a set groupoid $G \rightrightarrows M$ and any map $f\colon M' \to M$. Let $M' * G * M'$ denote the set of all $(y', g, x') \in M' \times G \times M'$ such that $f(y') = \beta g$, $\alpha g = f(x')$. Then there is an evident groupoid structure defined on $M' * G * M'$ by

$$\beta'(y',g,x') = y', \quad \alpha'(y',g,x') = x', \quad (z',h,y')(y',g,x') = (z',hg,x'),$$
$$1'_{x'} = (x', 1_{f(x')}, x'), \qquad (y',g,x')^{-1} = (x',g^{-1},y').$$

We denote this by $f^{\downdownarrows}G$ and call it the *pullback (set) groupoid of G over f.*

If G is a group, so that f is a constant map, then $f^{\downdownarrows}G$ is the trivial groupoid $M' \times G \times M'$.

If G is now a Lie groupoid and f a smooth map, we clearly need to assume that $f \times f\colon M' \times M' \to M \times M$ and $(\beta,\alpha)\colon G \to M \times M$ are transversal, so that the pullback manifold in Figure 2.1 exists. If G is locally trivial, it is easily seen that the pullback groupoid structure on $f^{\downdownarrows}G$ makes it a Lie groupoid, also locally trivial, with the appropriate universal property. However in general the transversality condition is

$$\begin{array}{ccc} f^{\downdownarrows}G & \longrightarrow & G \\ \Big\downarrow & & \Big\downarrow \, (\beta,\alpha) \\ M'\times M' & \underset{f\times f}{\longrightarrow} & M\times M \end{array}$$

Fig. 2.1.

not sufficient, since the target projection $f^{\downdownarrows}G \to M'$ may not be a submersion. The additional condition incorporated in the following result is often available.

Proposition 2.3.1 *Let $G \rightrightarrows M$ be a Lie groupoid and let $f\colon M' \to M$ be a smooth map. Suppose that the pullback manifold of $(\beta,\alpha)\colon G \to M\times M$ across the Cartesian square $f\times f\colon M'\times M' \to M\times M$ exists, and that the composition $\overline{\beta}\colon f^{!}G \to M$, $(g,x') \mapsto \beta g$, is a surjective submersion. Then with the pullback manifold structure and the groupoid structure above, $f^{\downdownarrows}G$ is a Lie groupoid on M' and the natural projection $f^{\downdownarrows}\colon f^{\downdownarrows}G \to G$ is a morphism of Lie groupoids over f.*

Further, if $F\colon G' \to G$, $f\colon M' \to M$ is a morphism of Lie groupoids, then $F = f^{\downdownarrows}\circ F^{\downdownarrows}$ where $F^{\downdownarrows}\colon G' \to f^{\downdownarrows}(G)$ is $g' \mapsto (\beta' g', \varphi(g'), \alpha' g')$.

With this structure, $f^{\downdownarrows}G$ is the *pullback Lie groupoid of G over f.*

Proof Construct the manifold $f^{\downdownarrows}G$ as the pullback

$$\begin{array}{ccc} f^{\downdownarrows}G & \longrightarrow & f^{!}G \\ \Big\downarrow & & \Big\downarrow \, \overline{\beta} \\ M' & \underset{f}{\longrightarrow} & M. \end{array}$$

Since $\overline{\beta}$ is a surjective submersion by assumption, it follows that the projection $f^{\downdownarrows}G \to M'$ is also, and this is the target projection of $f^{\downdownarrows}G$. Now it is straightforward to see that inversion is a diffeomorphism and that the composition is smooth. The factorizability condition is immediate. □

If f is a surjective submersion, then the projection $f^{!}G \to G$, and hence $\overline{\beta}$, is a surjective submersion, and so the additional condition above holds. This case, and the case in which G is locally trivial, are sufficient for most purposes.

In principle, 2.3.1 allows the study of arbitrary morphisms to be reduced to the base–preserving case. This will be useful when we come to consider morphisms of Lie algebroids.

Example 2.3.2 The frame groupoid $\Phi(f^{!}E)$ of a pullback vector bundle $f^{!}E$ is the pullback $f^{\Downarrow}\Phi(E)$ of the frame groupoid. ⊠

For pullbacks of general diagrams of Lie groupoids, see 2.4.14.

2.4 General quotients and fibrations

For set groupoids and Lie groupoids alike, the description of general groupoid morphisms involves phenomena similar to those which arose in the case of vector bundles in §2.1.

Certain problems arise already in the purely algebraic case. Firstly, the image of a groupoid morphism need not be a subgroupoid; it may happen that a product $F(h)F(g)$ is defined but the product hg is not and that another pair h_1, g_1 with $F(h_1) = F(h)$, $F(g_1) = F(g)$ and h_1g_1 defined cannot be found. This can occur even for morphisms of trivial groupoids, such as that in Example 2.2.6.

Secondly, the usual concept of kernel, as applied to groupoid morphisms, does not adequately measure injectivity.

Definition 2.4.1 Let $F\colon\ G \to G'$, $f\colon\ M \to M'$, be a morphism of groupoids. Then the *kernel* of (F, f) is the set

$$\{g \in G \mid F(g) = 1_{x'},\ \exists x' \in M'\}.$$

□

If F is a surjective submersion, then the kernel is a closed embedded submanifold of the domain.

Example 2.4.2 Let $P(M, G, \pi)$ be a principal bundle and consider the gauge groupoid $\Omega = \frac{P\times P}{G}$ constructed in §1.3. It is easy to see that the map $F\colon\ P\times P \to \Omega$, $(u_2, u_1) \mapsto \langle u_2, u_1\rangle$, is a morphism of Lie groupoids over $\pi\colon\ P \to M$, where $P \times P$ has the pair groupoid structure of 1.1.7.

The kernel is the diagonal Δ_P of P, which is the base subgroupoid of $P \times P$. ⊠

This example shows that surjective groupoid morphisms are not determined by their kernels: both the morphism in 2.4.2 and $\mathrm{id}_{P\times P}$ are surjective morphisms with kernel Δ_P.

This section is devoted to extending the notion of kernel so as to restore a form of the familiar 'first isomophism theorem'. A notion of kernel, however generalized, must be expressed in terms of data on the domain. We are accordingly seeking a general class of groupoid morphisms which are determined, up to isomorphism, by data on their domains. There are several candidates, the most inclusive of which is the class of fibrations.

Definition 2.4.3 Let $F\colon G \to G'$, $f\colon M \to M'$ be a morphism of Lie groupoids. Then (F, f) is a *fibration* if both f and $F^!\colon G \to f^!G'$ are surjective submersions. □

Fibrations are closed under composition.

The set fibration condition is precisely what is needed to guarantee that given any elements h, g of G such that $F(h)F(g)$ is defined, there exists $h_1 \in G$ such that $h_1 g$ is defined and $F(h_1) = F(h)$; then

$$F(h)F(g) = F(h_1)F(g) = F(h_1 g)$$

shows that the product $F(h)F(g)$ is determined by the domain groupoid and the map.

The surjective submersion condition on $F^!$ ensures that the kernel K of F is a Lie subgroupoid of G. Regard K as the preimage of

$$\{(m, 1'_{f(m)}) \mid m \in M\} \subseteq f^!G'$$

under $F^!$. Then $\alpha\colon K \to M$ corresponds to the restriction of $F^!$ to the complete inverse image K. It follows that $\alpha\colon K \to M$ is a surjective submersion and K is a Lie subgroupoid of G.

Before proceeding we need a lemma. Call the commutative square of smooth maps in Figure 2.2(a) *versal* if the pullback $M \times_{M'} B'$ exists (as a submanifold of the product) and the induced map $B \to M \times_{M'} B'$ is a surjective submersion. In this section we use pr_f to denote the projection $R(f) \to M$, $(y, x) \mapsto x$.

Lemma 2.4.4 *Given a commutative diagram as in Figure* 2.2(a) *in*

$$\begin{array}{ccc} B & \xrightarrow{F} & B' \\ p \downarrow & & \downarrow p' \\ M & \xrightarrow[f]{} & M' \end{array} \qquad\qquad \begin{array}{ccc} R(F) & \xrightarrow{\mathrm{pr}_F} & B \\ p\times p \downarrow & & \downarrow p \\ R(f) & \xrightarrow[\mathrm{pr}_f]{} & M \end{array}$$

(a) (b)

Fig. 2.2.

which each of the maps F, f, p and p' is a surjective submersion, the diagram is versal if and only if the diagram of Figure 2.2(b) *is versal. When this is so the restriction of $p \times p$ to $R(F) \to R(f)$ is a surjective submersion.*

Proof The conditions on p, F and f ensure that $R(F)$ and $R(f)$ are closed embedded submanifolds of $B \times B$ and $M \times M$, with pr_F and pr_f surjective submersions, and that the pullbacks $M \times_{M'} B'$ and $F(f) \times_M B$ exist. On the set level, the result is easy: if $R(F) \to R(f) \times_M B$ is surjective then, given $(m, b') \in M \times_{M'} B'$ there is $b \in B$ with $F(b) = b'$ and the element $((m, pb), b)$ of $R(f) \times_M B$ lifts to some $(a, b) \in R(F)$; now $a \in B$ is a lift of (m, b') as required.

Conversely, if $B \to M \times_{M'} B'$ is surjective then, given $((n, m), b) \in R(f) \times_M B$ we have $pb = m$ and so $(n, F(b))$ lies in $M \times_{M'} B'$ and therefore has a lift $a \in B$; now (a, b) lies in $R(F)$ and is a lift of $((n, m), b)$ as required. Lastly, if $R(F) \to R(f) \times_M B$ and $p\colon B \to M$ are surjective, it is clear that $p \times p\colon R(F) \to R(f)$ is surjective also.

$$\begin{array}{ccc} T_{(a,b)}(R(F)) & \xrightarrow{T_{(a,b)}(\mathrm{pr}_F)} & T_b(B) \\ \downarrow & & \downarrow \\ T_{(m,n)}(R(f)) & \xrightarrow[T_{(m,n)}(\mathrm{pr}_f)]{} & T_n(M) \end{array}$$

Fig. 2.3.

Now for each $(a, b) \in R(F)$ there are corresponding diagrams of linear

$$\begin{array}{ccc} T_b(B) & \xrightarrow{T_b(F)} & T_{F(b)}(B') \\ \big\downarrow & & \big\downarrow \\ T_n(M) & \xrightarrow[T_n(f)]{} & T_{f(n)}(M') \end{array}$$

Fig. 2.4.

maps as shown in Figures 2.3 and 2.4, where $m = pa$, $n = pb$. Here

$$T_{(a,b)}(R(F)) = \{(Y, X) \in T_a(B) \times T_b(B) \mid T(F)(Y) = T(F)(X)\},$$

with a similar equation for $T_{(m,n)}(R(f))$. The same argument applied on the tangent level then completes the proof. □

We proceed to the notion of kernel system via an intermediate notion of congruence.

Definition 2.4.5 Let G be a Lie groupoid on M. A *(smooth) congruence* on G is a pair (S, R) where $S \subseteq G \times G$ and $R \subseteq M \times M$ satisfy the following conditions:

(C1) S is a closed, embedded wide Lie subgroupoid of the pair groupoid $G \times G$ on G and R is a closed, embedded wide Lie subgroupoid of the pair groupoid $M \times M$ on M;

(C2) S is a Lie subgroupoid with base R of the Cartesian product groupoid $G \times G$ on $M \times M$;

(C3) the square in (4) below is versal.

$$\begin{array}{ccc} S & \xrightarrow{\mathrm{pr}_S} & G \\ {\scriptstyle \alpha\times\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ R & \xrightarrow[\mathrm{pr}_R]{} & M \end{array} \tag{4}$$

□

Theorem 2.4.6 *If $F\colon G \to G'$, $f\colon M \to M'$, is a fibration of Lie groupoids, then $(R(F), R(f))$ is a congruence on G. Conversely, given a Lie groupoid G on M and a congruence (S, R) on G, there is a unique*

Lie groupoid structure on the quotient sets $G' = G/S$, $M' = M/R$ such that the natural projections $F\colon G \to G'$, $f\colon M \to M'$ form a morphism of Lie groupoids. Further, this (F, f) is a fibration and is universal for morphisms $\Phi\colon G \to H$, $\varphi\colon M \to N$ such that $\Phi \times \Phi$ maps S to the diagonal of H and $\varphi \times \varphi$ maps R to the diagonal of N.

Proof Assume $F\colon G \to G'$, $f\colon M \to M'$, is a fibration of Lie groupoids. Then $R(F)$ and $R(f)$ satisfy (C1) of 2.4.5, and 2.4.4 applied to F, f, pr_F and pr_f yields (C3) and the fact that $\alpha\times\alpha\colon R(F) \to R(f)$ is a surjective submersion. The remaining conditions in (C2) then follow easily.

Conversely, consider a congruence (S, R) on a Lie groupoid $G \rightrightarrows M$. By (C1) and the Godement criterion, there are manifold structures on the quotient sets $G' = G/S$, $M' = M/R$ such that the natural projections $F\colon G \to G'$, $f\colon M \to M'$ are surjective submersions. Define $\alpha'\colon G' \to M'$ by $\alpha' \circ F = f \circ \alpha$; since α, F and f are surjective submersions, it follows that α' is smooth and itself a surjective submersion. Likewise define β' by $\beta' \circ F = f \circ \beta$. Given $g', h' \in G'$ with $\alpha'(g') = \beta'(h')$, choose any $h \in G$ with $F(h) = h'$ and then choose $g \in G$ with $F(g) = g'$ and $\alpha(g) = \beta(h)$; that this is possible follows from (C3) and 2.4.4. We can now define $g'h' = F(gh)$, for (C2) implies that the product is unambiguous and defines a groupoid structure on G' with base M'. Certainly $F\colon G \to G'$, $f\colon M \to M'$ is a morphism (of set groupoids) and, by (C3) and 2.4.4, the diagram

$$\begin{array}{ccc} G & \xrightarrow{\;F\;} & G' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ M & \xrightarrow[\;f\;]{} & M' \end{array}$$

is versal. It remains to prove that the groupoid structure in G' is Lie.

Let $\delta\colon G \times_\alpha G \to G$ denote the division $\delta(g, h) = gh^{-1}$ as in 1.2.6. To prove that the corresponding division map $\delta'\colon G' \times_{\alpha'} G' \to G'$ is smooth, it suffices to show that the restriction $F \times F\colon G \times_\alpha G \to G' \times_{\alpha'} G'$ is a surjective submersion, and this now follows by applying 2.4.4 to the diagram in Figure 2.5.

The universal property and the uniqueness are easily verified. □

Theorem 2.4.6 provides a complete characterization of fibrations of

$$\begin{array}{ccccc} G \times_\alpha G & \xrightarrow{\mathrm{pr}_\alpha} & G & \xrightarrow{\alpha} & M \\ {\scriptstyle F\times F}\downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f} \\ G' \times_{\alpha'} G' & \xrightarrow[\mathrm{pr}_{\alpha'}]{} & G' & \xrightarrow[\alpha']{} & M'. \end{array}$$

Fig. 2.5.

Lie groupoids in terms of data from the domain groupoid. We now reformulate this notion of congruence so that its relation to the usual notion of kernel can be seen clearly.

Consider a Lie groupoid $G \rightrightarrows M$ and any closed, embedded wide Lie subgroupoid N of G (not necessarily normal). Define

$$E = \{(g,h) \mid \alpha g = \alpha h,\ gh^{-1} \in N\} \subseteq G \times G.$$

Then $E = \delta^{-1}(N)$ where δ is the division map. Since $\alpha\colon G \to M$ and δ are surjective submersions, it follows that E is a closed, embedded submanifold of $G \times G$. Further,

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{pr}_E} & G \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \beta} \\ N & \xrightarrow[\alpha]{} & M \end{array}$$

is a pullback of manifolds, and since $\alpha\colon N \to M$ is a surjective submersion by assumption, it follows that pr_E is also. The Godement criterion now implies that the set of one–sided cosets

$$Ng = \{\nu g \mid \alpha\nu = \beta g,\ \nu \in N\}$$

as g ranges through G, which we denote $G \mathbin{\cdot^{\cdot^{\cdot}}} N$, has a smooth manifold structure such that $q\colon G \to G \mathbin{\cdot^{\cdot^{\cdot}}} N$, $g \mapsto Ng$, is a surjective submersion. Note that $G \mathbin{\cdot^{\cdot^{\cdot}}} N$ is usually not a groupoid, even if N is normal, but that $\alpha\colon G \to M$ quotients to a well–defined surjective submersion $G \mathbin{\cdot^{\cdot^{\cdot}}} N \to M$, which we also denote by α.

Definition 2.4.7 A *normal subgroupoid system* in $G \rightrightarrows M$ is a triple

$\mathscr{N} = (N, R, \theta)$ where N is a closed, embedded, wide Lie subgroupoid of G, where R is a closed, embedded, wide, Lie subgroupoid of the pair groupoid $M \times M$, and θ is an action of R on the map $\alpha\colon G \mathbin{.\!\cdot^{\bullet}} N \to M$ just described, such that the following conditions hold.

(N1) Consider $(n, m) \in R$ and $Ng \in G \mathbin{.\!\cdot^{\bullet}} N$ with $\alpha(Ng) = m$. Then, writing $\theta(n, m)(Ng) = Nh$, we have $(\beta h, \beta g) \in R$.

(N2) For $(n, m) \in R$ we have $\theta(n, m)(N1_m) = N1_n$.

(N3) Consider $(n, m) \in R$ and $Ng \in G \mathbin{.\!\cdot^{\bullet}} N$ with $\alpha(Ng) = m$, and $h \in G$ with $\alpha h = \beta g$. Then if

$$\theta(n, m)(Ng) = Ng_1 \qquad \text{and} \qquad \theta(\beta g_1, \beta g)(Nh) = Nh_1,$$

we have $\theta(n, m)(Nhg) = Nh_1g_1$. □

For $\nu \in N$ it follows from (N1) that $(\beta\nu, \alpha\nu) \in R$ and it then follows from (N2) and (N3) that if $Ng \in G \mathbin{.\!\cdot^{\bullet}} N$ has $\alpha(Ng) = \alpha\nu$, then $\theta(\beta\nu, \alpha\nu)(Ng) = Ng\nu^{-1}$. In particular, if $\nu \in IN$, $g \in G$ and $g\nu g^{-1}$ is defined, then $g\nu g^{-1} \in N$.

Theorem 2.4.8 (i) *Let $\mathscr{N} = (N, R, \theta)$ be a normal subgroupoid system on G, and define*

$$S = \{(h, g) \in G \times G \mid (\alpha h, \alpha g) \in R \text{ and } \theta(\alpha h, \alpha g)(Ng) = Nh\}.$$

Then (S, R) is a congruence on G.

(ii) *Let (S, R) be a congruence on G. Then $N = \{g \in G \mid (g, 1_{\alpha g}) \in S\}$ is a closed, embedded, Lie subgroupoid of G. Define an action θ of R on $G \mathbin{.\!\cdot^{\bullet}} N$ by*

$$\theta(n, m)(Ng) = Nh,$$

where $m = \alpha g$ and h is any element of G with $\alpha h = n$ and $(h, g) \in S$. Then (N, R, θ) is a normal subgroupoid system on G.

These two constructions are mutually inverse.

Proof (i) The algebraic conditions are easily verified. The smoothness conditions reduce to proving that three maps are surjective submersions, namely $\mathrm{pr}_S\colon S \to G$, $\alpha \times \alpha\colon S \to R$ and the map

$$S \to R \times_\alpha G, \qquad (h, g) \mapsto ((\alpha h, \alpha g), g)$$

of (C3). The first two of these follow easily from the last. To prove the last, consider the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\;\subseteq\;} & G \times G \\
\big\downarrow & & \big\downarrow \quad q \times \mathrm{id} \\
R \times_\alpha G & \xrightarrow[\widehat{\theta}]{} & (G \mathbin{.^{\cdot^{\cdot}}} N) \times G
\end{array}
$$

where $\widehat{\theta}((n,m),g) = (\theta(n,m)(Ng), g)$. One proves that this is a pullback of manifolds by identifying $(((n,m),g),(h',g'))$, where $(n,m) \in R$, $n = \alpha g$, $g = g'$, and $\theta(n,m)(Ng) = Nh'$, with $(h',g') \in S$. It then follows that because $q \times \mathrm{id}$ is a surjective submersion, the desired map $S \to R \times_\alpha G$ is also.

(ii) First observe that $\{((m,m),1_m) \mid m \in M\}$ is a closed embedded submanifold of $R \times_\alpha G$ and so its preimage under $S \to R \times_\alpha G$, namely $L = \{(h,g) \in S \mid g = 1_{\alpha h}\}$, is a closed embedded submanifold of S. By considering

$$G \to G \times_\alpha G, \qquad g \mapsto (g, 1_{\alpha g}),$$

it follows that N is a closed embedded submanifold of G and that $\alpha\colon N \to M$, which corresponds to the restriction of $S \to R \times_\alpha G$ to $L \to M$, is a surjective submersion. Thus N is also a Lie subgroupoid of G.

That θ is well–defined and an action follow from the algebraic conditions in (C1) – (C3). To prove that it is smooth consider the diagram

$$
\begin{array}{ccccc}
 & & S & & \\
 & \swarrow & & \searrow & \\
 & R \times_\alpha G & & G & \\
\mathrm{id} \times_\alpha q & \big\downarrow & & \big\downarrow & q \\
 & R \times_\alpha (G \mathbin{.^{\cdot^{\cdot}}} N) & \xrightarrow[\theta]{} & (G \mathbin{.^{\cdot^{\cdot}}} N) &
\end{array}
$$

in which $S \to G$ is $(h,g) \mapsto h$ and $S \to R \times_\alpha G$ is $(h,g) \mapsto ((\alpha h, \alpha g), g)$. It is easily proved by a direct calculation with tangent vectors that $\mathrm{id} \times_\alpha q$ is a surjective submersion, and since $S \to R \times_\alpha G$ is also, the smoothness of θ follows from that of $S \to G \mathbin{.^{\cdot^{\cdot}}} N$.

That the two constructions are mutually inverse is easily verified. □

Putting 2.4.6 and 2.4.8 together, we obtain a bijective correspondence between normal subgroupoid systems of G and fibrations of Lie groupoids with domain G.

Given a fibration $F\colon G \to G'$, $f\colon M \to M'$, let K be the kernel of F as in 2.4.1. Then $F^{!}$ induces a diffeomorphism $\overline{F}\colon G \,.\!\cdot^{\!\cdot} K \to f^{!}G'$ and we use $\overline{F}$ to transport the canonical action $\theta_0(n,m)(m,g') = (n,g')$ of $R(f)$ on $f^{!}G' \to M$ to an action θ of $R(f)$ on $\alpha\colon G \,.\!\cdot^{\!\cdot} K \to M$. It is easy to check that $(K, R(f), \theta)$ is the normal subgroupoid system corresponding to $(R(F), R(f))$ under 2.4.8; we call it the *kernel system of* (F, f). Note that this notion of kernel system is not defined for arbitrary morphisms.

Conversely, let $\mathscr{N} = (N, R, \theta)$ be a normal subgroupoid system on G. Then $R = R(f)$ for some surjective submersion $f\colon M \to M'$. Let G' denote the set of orbits of $G \,.\!\cdot^{\!\cdot} N$ under θ, with $\langle Ng \rangle$ denoting the orbit of Ng. Write $F\colon G \to G'$ for $F(g) = \langle Ng \rangle$. One may prove that G' has a quotient manifold structure, by the methods of 2.4.6 and 2.4.8; see also 2.1.2. Define groupoid projections on G', with base M', by $\alpha' \circ F = f \circ \alpha$, $\beta' \circ F = f \circ \beta$. Define composition for $\langle Nh \rangle, \langle Ng \rangle \in G'$ with $\alpha'(\langle Nh \rangle) = \beta'(\langle Ng \rangle)$ by

$$\langle Nh \rangle \langle Ng \rangle = \langle Nh_1 g \rangle$$

where h_1 is any element of G with $Nh_1 = \theta(\beta g, \alpha h)(Nh)$. One can prove directly now that G' is a Lie groupoid on M' with (F, f) a fibration, and that G' is the Lie groupoid which corresponds under 2.4.6 to the congruence on G defined by $\mathscr{N}$. We write $G' = G/\mathscr{N}$ and call it the *quotient Lie groupoid of G over the normal subgroupoid system $\mathscr{N}$*. We call (F, f) the *quotient fibration corresponding to $\mathscr{N}$*. The universal property for F is as follows: the proof is immediate.

Theorem 2.4.9 *Let $F\colon G \to G'$, $f\colon M \to M'$, be a fibration of Lie groupoids, with kernel system $\mathscr{K} = (K, R(f), \theta)$.*

Suppose that $\Phi\colon G \to H$ is any morphism of Lie groupoids over a smooth map $\varphi\colon M \to P$ which annuls *$\mathscr{K}$ in the sense that:*

(i) *for all $k \in K$, $\Phi(k)$ is an identity of H;*

(ii) *$\varphi \times \varphi$ maps $R(f)$ into the diagonal of $P \times P$;*

(iii) *if $\overline{\Phi}$ is the induced map $G \,.\!\cdot^{\!\cdot} K \to H$, then*

$$\overline{\Phi}(\theta(n,m)(Kg)) = \overline{\Phi}(Kg)$$

for all $(n,m) \in R(f)$ *and* $Kg \in G \mathbin{.\!\cdot^{\bullet}} K$ *with* $\alpha g = m$.

Then there is a unique morphism of Lie groupoids $\Psi\colon G' \to H$ *over* $\psi\colon M' \to P$ *such that* $\Psi \circ F = \Phi$ *and* $\psi \circ f = \varphi$. *In particular, if* (Φ, φ) *is a fibration and* $\mathscr{K}$ *is its kernel system, then* (Ψ, ψ) *is an isomorphism of Lie groupoids.*

Example 2.4.10 Let $P(M, G, \pi)$ be a principal bundle. Continuing 2.4.2, the map $F\colon P \times P \to \Omega$ is a fibration over π. The kernel pair $R(\pi) = P \times_\pi P$ acts on $P \times P$ by

$$(ug, u)\cdot(v, u) = (vg, ug).$$

The orbits of this action coincide with the orbits of the diagonal G action on $P \times P$ and the quotient is indeed the gauge groupoid. ⊠

Example 2.4.11 If N is a normal subgroupoid of $\mathscr{I}G$ in the sense of §2.2 then taking $R = \Delta$ and θ to be the trivial representation gives a normal subgroupoid system. ⊠

Example 2.4.12 The projection $p_G\colon TG \to G$, for any Lie groupoid $G \rightrightarrows M$, is a fibration over $p_M\colon TM \to M$. Elements of the conventional kernel $K = T_1G$ are of the form $X + T(1)(x)$ where $X \in AG$, $x \in TM$ lie over the same point of M. (We are using the basic properties of the Lie algebroid, for which see §3.5.) Applying 1.4.14 we have

$$(Y + T(1)(y)) \bullet (X + T(1)(x)) = Y + X + T(1)(x)$$

where $y = aX + x$. So T_1G is the action Lie groupoid $A_{\mathrm{vb}}G \lessdot TM$ where $A_{\mathrm{vb}}G$ denotes the additive groupoid defined by the vector bundle structure of AG and the action is $(X, x) \mapsto aX + x$.

Given $\xi \in T_gG$, applying 1.4.14 again, the coset defined by ξ consists of all elements of the form

$$(Y + T(1)(y)) \bullet \xi = \xi + \overrightarrow{Y}(g)$$

where $y = T(\beta)(\xi)$ and $Y \in A_{\beta g}G$. It is now clear directly that the map induced by p_G, which maps the coset $\langle\xi\rangle$ to $(T(\alpha)(\xi), g) \in p_M^{!}G$, is a diffeomorphism. There is no canonical way to describe the action of $R(p_M) = TM \oplus TM$ on $TG \mathbin{.\!\cdot^{\bullet}} T_1G$ in terms of representatives. ⊠

The notion of fibration arises naturally in the context of direct products and general pullbacks in the category of Lie groupoids.

Example 2.4.13 For Lie groupoids $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$ there is a natural Cartesian product Lie groupoid $G_1 \times G_2 \rightrightarrows M_1 \times M_2$, and the projections $G_1 \times G_2 \to G_1$ and $G_1 \times G_2 \to G_2$ are fibrations. ⊠

For pullbacks of general diagrams of Lie groupoid morphisms, it is natural to assume that the pullback of the base manifolds exists.

Proposition 2.4.14 *Let $F_1\colon G_1 \to G$ and $F_2\colon G_2 \to G$ be fibrations of Lie groupoids over $f_1\colon M_1 \to M$ and $f_2\colon M_2 \to M$. Suppose that the pullback manifold $\overline{M}$ of f_1 and f_2 exists. Then the pullback manifold $\overline{G}$ of F_1 and F_2 exists and is an embedded Lie subgroupoid of the Cartesian product groupoid $G_1 \times G_2$. Furthermore, $\overline{G}$ is the pullback of F_1 and F_2 in the category of Lie groupoids.*

Proof The algebraic requirements are easily verified. The smoothness considerations will follow easily once it is established that the source map $\overline{\alpha}\colon \overline{G} \to \overline{M}$ is a surjective submersion.

Let f denote the diagonal map $\overline{M} \to M$, $(m_1, m_2) \mapsto f_1(m_1) = f_2(m_2)$, and observe that

$$f^{!}G \to f_1^{!}G \times f_2^{!}G,\ ((m_1, m_2), g) \mapsto ((m_1, g), (m_2, g))$$

represents $f^{!}G$ as an embedded submanifold of $f_1^{!}G \times f_2^{!}G$. Let F denote the map $F_1^{!} \times F_2^{!}\colon G_1 \times G_2 \to f_1^{!}G \times f_2^{!}G$. Then $\overline{G}$ is precisely $F^{-1}(f^{!}G)$ and since F is a surjective submersion, it follows that $\overline{G}$ is an embedded submanifold of $G_1 \times G_2$, and that the restriction

$$\overline{G} \to f^{!}G, \qquad (g_1, g_2) \mapsto ((\alpha_1 g_1, \alpha_2 g_2), F_1(g_1))$$

is still a surjective submersion. So it suffices to prove that the projection $f^{!}G \to \overline{M}$ is a surjective submersion. Since it is the left–hand side of the pullback diagram

$$\begin{array}{ccc} f^{!}G & \longrightarrow & G \\ \downarrow & & \downarrow \alpha \\ \overline{M} & \longrightarrow & M \end{array}$$

and α is a surjective submersion, it follows that $f^{!}G \to \overline{M}$ is also. □

* * * * *

If $F\colon G \to G'$, $f\colon M \to M'$, is a fibration of Lie groupoids, corresponding to a normal subgroupoid system $\mathscr{N} = (N, R(f), \theta)$ for which N consists entirely of identity elements, then $G \mathbin{\iddots} N = G$ and the diffeomorphism between $G \mathbin{\iddots} N$ and $f^{!}G'$ can be identified with $F^{!}$. Thus F is an action morphism corresponding to an action of G' on f.

We now consider the other extreme, in which both $R(f)$ and θ are determined by N. This case includes quotients which preserve the base, and in general behaves in a more familiar way.

Definition 2.4.15 Let $G \rightrightarrows M$ be a Lie groupoid.

(i) A normal subgroupoid system $\mathscr{N} = (N, R, \theta)$ on G is *uniform* if the anchor of G, restricted to $N \to R$, is a surjective submersion.

(ii) A congruence (S, R) on G is *uniform* if the map from S to

$$(R \times R) \times_{M \times M} G = \{(n_2, n_1, m_2, m_1, g) \in R \times R \times G \mid n_1 = \beta g,\ m_1 = \alpha g\}$$

given by $(h, g) \mapsto (\beta h, \beta g, \alpha h, \alpha g, g)$, is a surjective submersion.

(iii) A morphism of Lie groupoids $F\colon G \to G', f\colon M \to M'$, is a *uniform fibration* if both f and $F^{\downdownarrows}\colon G \to f^{\downdownarrows}G'$ are surjective submersions. (Here $f^{\downdownarrows}G$ is the pullback groupoid of §2.3.) □

If $\mathscr{N}$ is uniform, then θ is entirely determined by the fact that

$$\theta(\beta\nu, \alpha\nu)(Ng) = Ng\nu^{-1}$$

for $\nu \in N$ with $\alpha g = \alpha\nu$. Thus N determines θ. Since N also determines R, as the image of its anchor, both $\mathscr{N}$ and $G/\mathscr{N}$ are entirely determined by N.

$$\begin{array}{ccc} N & \longrightarrow & S \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \chi \times \chi} \\ R & \longrightarrow & R \times R \end{array}$$

Fig. 2.6.

Theorem 2.4.16 *Let $\mathscr{N} = (N, R, \theta)$ be a normal subgroupoid system on a Lie groupoid G, with $F\colon G \to G' = G/\mathscr{N}$, $f\colon M \to M'$, the quotient fibration, and corresponding congruence (S, R).*

Then $\mathscr{N}$ is uniform if and only if (S, R) is uniform, and this is so if and only if (F, f) is a uniform fibration.

Proof Suppose that (F, f) is a uniform fibration. Then we apply 2.4.4 to

$$\begin{array}{ccccc}
S = R(F) & \longrightarrow & G & \xrightarrow{\quad F \quad} & G' \\
{\scriptstyle \chi\times\chi}\big\downarrow & & \big\downarrow{\scriptstyle \chi} & & \big\downarrow{\scriptstyle \chi'} \\
R \times R = R(f \times f) & \longrightarrow & M \times M & \xrightarrow[\quad f \times f \quad]{} & M' \times M'
\end{array}$$

with a minor modification since the anchors χ and χ' of G and G' need not be surjective submersions. It follows that (S, R) is uniform.

Next, if (S, R) is uniform then the diagram in Figure 2.6 is a pullback, where $N \to S$ is $\nu \mapsto (\nu, 1_{\alpha\nu})$ and $R \to R\times R$ is $(n, m) \mapsto (n, m, m, m)$. It follows that $\chi\colon\ N \to R$ is a surjective submersion, so $\mathscr{N}$ is uniform.

The proof that uniformity of $\mathscr{N}$ implies uniformity of (F, f) can be carried out by using 2.4.4. □

2.5 General semidirect products

We come now to consider actions of one groupoid on another in such a way that structure is preserved. This requires a more elaborate setting than that of §1.6.

Definition 2.5.1 Let $G \rightrightarrows M$ and $W \rightrightarrows M'$ be Lie groupoids and let $f\colon M' \to M$ be a surjective submersion such that $f \circ \alpha_W = f \circ \beta_W\colon W \to M$. Write $p = f \circ \alpha_W$. Then G *acts smoothly on the Lie groupoid W via f* if there is given a smooth map $(g, w) \mapsto gw$ from $G * W = \{(g, w) \in G \times W \mid \alpha g = pw\}$ to W such that:

- $p(gw) = \beta g$ for all $(g, w) \in G * W$;
- for each $g \in G$, the map $w \mapsto gw$, $p^{-1}(\alpha g) \to p^{-1}(\beta g)$, is an isomorphism of Lie groupoids;
- $h(gw) = (hg)w$ whenever both hg and gw are defined;
- $1_{pw}w = w$ for all $w \in W$.

□

We will use α, β for the source and target on both G and W unless there is a real likelihood of confusion. Note that the condition $f \circ \alpha_W = f \circ \beta_W = p$ on f is the condition that (p, f) is a groupoid morphism from W to the base groupoid 1_M. In particular, f is constant on each transitivity component of W and if W is transitive, G must be a Lie group.

The requirement that f (or equivalently p) be a surjective submersion ensures that each $p^{-1}(m)$, $m \in M$, is a closed embedded Lie subgroupoid of W on base $f^{-1}(m)$.

Consider now an action $G * W \to W$ as in 2.5.1. There is an action of G on the map $f\colon M' \to M$ in the sense of §1.6 determined by

$$1'_{gm'} = g1'_{m'}.$$

The diffeomorphism $f^{-1}(\alpha g) \to f^{-1}(\beta g)$ induced by $g \in G$ is the base map corresponding to the isomorphism $p^{-1}(\alpha g) \to p^{-1}(\beta g)$ induced by g. In particular, the source and target projections $W \to M'$ are G–equivariant.

We now define a groupoid structure on $G * W$ with base M'. The source and target maps are

$$\alpha'(g, w) = \alpha(w), \qquad \beta'(g, w) = g\beta(w),$$

and the composition is

$$(g_2, w_2)(g_1, w_1) = (g_2 g_1, (g_1^{-1} w_2) w_1),$$

defined if $\alpha(w_2) = g_1 \beta(w_1)$. The identity element corresponding to m' is $(1_{f(m')}, 1'_{m'})$ and the inverse of (g, w) is $(g^{-1}, g(w^{-1}))$. It is routine to verify that this is a Lie groupoid structure on $G * W$. We denote it by $G \sqsubset\!\!\sqsupset W$ and call it the *general semidirect product*.

The construction clearly includes the usual semidirect product arising from a Lie group action on another Lie group by Lie group automorphisms and also, when W is a base groupoid, the action groupoid of §1.6.

There are natural morphisms $\iota\colon W \to G \sqsubset\!\!\sqsupset W$, $w \mapsto (1_{pw}, w)$ and $\pi\colon G \sqsubset\!\!\sqsupset W \to G$, $(g, w) \mapsto g$. Here ι is a morphism over M' and π is a fibration over f. One expects π to split in some sense, but it is clear that π cannot have a true right–inverse in general. What does exist is a morphism

$$\sigma_0\colon G \lessdot M' \to G \sqsubset\!\!\sqsupset W, \ (g, m') \mapsto (g, 1'_{m'})$$

over M', which is right–inverse to both $G \ltimes W \to G \lessdot M'$, $(g, w) \mapsto (g, \alpha(w))$ and $G \ltimes W \to G \lessdot M'$, $(g, w) \mapsto (g, \beta(w))$. The first of these is $\pi^!$ in the notation of 2.4.3; we temporarily denote the second by π'. It should be noted that, although both $G \ltimes W$ and $G \lessdot M'$ have groupoid structures, neither $\pi^!$ nor π' is usually a morphism. Nonetheless the image of σ_0 is a Lie subgroupoid of $G \ltimes W$ naturally isomorphic to $G \lessdot M'$ and is a complement to the normal subgroupoid $\iota(W)$ in the sense that each element of $G \ltimes W$ is uniquely a product $\mu\nu$ where $\mu \in \sigma_0(G \lessdot M')$ and $\nu \in \iota(W)$.

Definition 2.5.2 A *split fibration* of Lie groupoids consists of a fibration $F\colon G' \to G$, $f\colon M' \to M$, together with an action of G on f and a morphism $\sigma\colon G \lessdot M' \to G'$ which is right–inverse to $F^!$. □

Theorem 2.5.3 (i) *Let a Lie groupoid $G \rightrightarrows M$ act on a Lie groupoid $W \rightrightarrows M'$ via a map $f\colon M' \to M$. Then $\pi\colon G \ltimes W \to G$, $(g, w) \mapsto g$, together with the given action of G on M' and $\sigma_0\colon G \lessdot M' \to G \ltimes W$, $(g, m') \mapsto (g, 1'_{m'})$, is a split fibration.*

(ii) *Let $F\colon G' \to G$, $f\colon M' \to M$, be a split fibration with kernel W. Then there is an action of G on W, namely*

$$gw = \sigma(g, \beta'(w))w\,\sigma(g, \alpha'(w))^{-1},$$

and a unique isomorphism $\Psi\colon G \ltimes W \to G'$ such that $\Psi \circ \pi^! = F^!$ and $\Psi \circ \sigma_0 = \sigma$, namely $(g, w) \mapsto \sigma(g, \beta'(w))w$.

These constructions are mutually inverse, up to natural isomorphism.

We omit the proof, which uses techniques familiar from the preceding sections.

PBG–groupoids

The following description of extensions of Lie groupoids illustrates several of the constructions of this chapter. This material will be used in Chapters 6 and 8.

Consider an extension of locally trivial Lie groupoids

$$\Lambda \rightarrowtail \Phi \xrightarrow{\pi} \Omega. \tag{5}$$

Choose $m \in M$ and let $Q(M, H, q)$ and $P(M, G, p)$ denote the vertex bundles of Φ and Ω. Write $N = \Lambda_m$. The corresponding extension of principal bundles is therefore

$$N \rightarrowtail Q(M,H) \twoheadrightarrow P(M,G). \tag{6}$$

In what follows we will identify Φ and Ω with the gauge groupoids of Q and P.

Now $Q(P,N,\pi)$ is itself a principal bundle. Form the gauge groupoid $\Upsilon = \frac{Q\times Q}{N}$. For clarity we denote elements of Υ by $\langle v_2, v_1\rangle_N$, elements of Φ by $\langle v_2, v_1\rangle_H$ and elements of Ω by $\langle u_2, u_1\rangle_G$.

The group G does not generally act on H, much less on Q. It does, however, act on Υ by

$$\langle v_2, v_1\rangle_N g = \langle v_2 h, v_1 h\rangle_N,$$

where h is any element of H with $\pi(h) = g$. This is an action of G on $\Upsilon \rightrightarrows P$ by Lie groupoid automorphisms over the given principal action $P \times G \to P$, and is an action in the sense of 2.5.1 via the constant map $M \to \{\cdot\}$. This action is free and it is easy to see that the quotient manifold Υ/G, with its quotient groupoid structure in the sense of §2.4, is isomorphic to Φ under the map induced by $\Upsilon \to \Phi$, $\langle v_2, v_1\rangle_N \mapsto \langle v_2, v_1\rangle_H$.

Furthermore, this map $\Upsilon \to \Phi$ is an action morphism over $p\colon P \to M$. The induced action of Φ on P is

$$\langle v_2, v_1\rangle_H \ (u) = \pi(v_2)g^{-1}$$

where g is determined by $\pi(v_1) = ug$. Thus Υ can be constructed directly from (5) and $P = \Omega_m$. The action of G in terms of $\Upsilon = \Phi \lessdot P$ is the canonical action $(\zeta, u)g = (\zeta, ug)$.

This process can be reversed.

Definition 2.5.4 Let $P(M,G,p)$ be a principal bundle. A *PBG–groupoid* on $P(M,G)$ is a locally trivial Lie groupoid $\Upsilon \rightrightarrows P$ together with a right action of G on Υ by Lie groupoid automorphisms over the principal action $P \times G \to P$. □

Notice that the action of G on Υ must necessarily be free.

Proposition 2.5.5 *Given a PBG–groupoid $\Upsilon \rightrightarrows P(M,G)$, the quotient manifold Υ/G exists.*

Proof Since the quotient is defined by a group action, it suffices to show that the graph

$$\Gamma' = \{(\upsilon, \upsilon g) \mid \upsilon \in \Upsilon,\ g \in G\}$$

is a closed submanifold of $\Upsilon \times \Upsilon$. Denote the anchor $\Upsilon \to P \times P$ by χ and write Γ for the graph of the diagonal action of G on $P \times P$. Then $\Gamma' \subseteq (\chi \times \chi)^{-1}(\Gamma)$. Since Γ is a closed submanifold of P^4 and $\chi \times \chi$ is a surjective submersion, it suffices to show that Γ' is a closed submanifold of $(\chi \times \chi)^{-1}(\Gamma)$.

Define $f\colon (\chi \times \chi)^{-1}(\Gamma) \to \mathscr{I}\Upsilon,\ (\upsilon, \upsilon') \mapsto \upsilon'(\upsilon g)^{-1}$, where g is determined by υ and υ'. Clearly, $f^{-1}(1_P) = \Gamma'$, so it suffices to show that f is a surjective submersion. Since Υ is locally trivial, the division map $\delta\colon \Upsilon \times_\alpha \Upsilon \to \Upsilon$ is a surjective submersion, and so its restriction to $\delta^{-1}(\mathscr{I}\Upsilon) \to \mathscr{I}\Upsilon$ is also. Now it is only necessary to incorporate the effect of translation by the group action. □

Consider a PBG–groupoid $\Upsilon \rightrightarrows P(M, G, p)$. Denote the quotient manifold Υ/G by Φ, and elements of Φ by $\|\upsilon\|$ for $\upsilon \in \Upsilon$. Then φ has a groupoid structure on base M with source and target maps

$$\beta(\|\upsilon\|) = p(\beta(\upsilon)), \qquad \alpha(\|\upsilon\|) = p(\alpha(\upsilon))$$

and multiplication of $\|\upsilon_2\|$ and $\|\upsilon_1\|$ with $\alpha(\|\upsilon_2\|) = \beta(\|\upsilon_1\|)$ defined by

$$\|\upsilon_2\|\,\|\upsilon_1\| = \|\upsilon_2(\upsilon_1 g)\|$$

where $g \in G$ is determined by $\alpha(\upsilon_2) = \beta(\upsilon_1)g$. A modification of the argument in §1.3 shows that Φ is a locally trivial Lie groupoid and the canonical map $\natural\colon \Upsilon \to \Phi,\ \upsilon \mapsto \|\upsilon\|$, is a morphism. Furthermore, $\natural$ is an action morphism.

The short exact sequence $\mathscr{I}\Upsilon \rightarrowtail \Upsilon \twoheadrightarrow P \times P$ now quotients to an extension of Lie groupoids

$$\frac{\mathscr{I}\Upsilon}{G} \rightarrowtail \Phi \twoheadrightarrow \frac{P \times P}{G}.$$

Thus extensions (5) of locally trivial Lie groupoids can be studied in terms of the single Lie groupoid $\Upsilon \rightrightarrows P$ and the action of G upon it. This correspondence emphasizes the significance of the distinction between automorphisms of a principal bundle and automorphisms of its gauge groupoid.

2.6 Classes of morphisms

Many of the general algebraic constructions possible for groupoids may be characterized in terms of the properties of associated morphisms. This is a peculiarity of groupoids (and of Lie algebroids, and other related concepts), with no precursor in the theory of groups.

The definitions which follow have been given in earlier sections, but are collected here for the purpose of comparison. For simplicity, we consider only morphisms over base–maps which are surjective submersions.

$$\begin{array}{ccc} f^{!}G & \longrightarrow & G \\ \downarrow & & \downarrow \;\alpha \\ M' & \underset{f}{\longrightarrow} & M. \end{array}$$

Fig. 2.7.

Consider a morphism of Lie groupoids $F\colon\ G' \to G,\ f\colon\ M' \to M$, and form the pullback manifold as in Figure 2.7. Denote by $F^{!}\colon\ G' \to f^{!}G$ the induced map $g' \mapsto (\alpha' g', F(g'))$.

Then:

- (F, f) is a *fibration* if $F^{!}$ is a surjective submersion;
- (F, f) is an *action morphism* if $F^{!}$ is a diffeomorphism.

Now form the pullback groupoid, as in §2.3,

$$\begin{array}{ccc} f^{\downarrow\downarrow}G & \longrightarrow & G \\ \downarrow & & \downarrow \;\chi \\ M' \times M' & \underset{f \times f}{\longrightarrow} & M \times M. \end{array}$$

There is an induced map $F^{\downarrow\downarrow}\colon\ G' \to f^{\downarrow\downarrow}G,\ g' \mapsto (\beta' g', F(g'), \alpha' g')$, and

- (F, f) is a *uniform fibration* if $F^{\downarrow\downarrow}$ is a surjective submersion,
- (F, f) is an *inductor* if $F^{\downarrow\downarrow}$ is a diffeomorphism.

These four classes of morphism may be regarded as giving four notions of surjectivity (or epimorphism) for groupoid morphisms. The fibrations are the largest class for which the kernel of 2.4.1 is a Lie subgroupoid of G'.

We saw in §1.6 that action morphisms correspond to, and characterize, actions of the target groupoid on the base map. In §2.4 of the present chapter we showed that fibrations correspond to, and characterize, quotients over general normal subgroupoid systems, and that uniform fibrations likewise correspond to quotients over a specific class of normal subgroupoid systems.

It is easy to see, in a similar way, that inductors correspond to, and characterize, pullback groupoids. They can be characterized amongst the fibrations as those uniform fibrations whose kernel may be identified with an equivalence relation on the base manifold; this equivalence relation is then the kernel pair $R(f)$, and the inductor is equivalent to the morphism $F^{\Downarrow}$.

2.7 Notes

§2.1. For the basic quotienting processes for manifolds, I follow [Serre, 1992, III§12]. For the basic facts about vector bundles see [Dieudonné, 1972] or [Greub et al., 1972].

The descent process given in this section is as used in [Higgins and Mackenzie, 1990a, §4]. I do not know of a book account of descent for C^∞ vector bundles but the material of this section is surely folklore.

§2.2. The results of this section on the base–preserving case are taken from [Mackenzie, 1987a, I§2]. For groupoids which are locally trivial, the smooth case is a simple analogue of base–preserving quotients for set groupoids, which were dealt with by Higgins [1971]. In [Mackenzie, 1987a], N was allowed to contain elements outside $\mathscr{I}G$; however, only the elements of $N \cap \mathscr{I}G$ were used in the definition or the quotienting process.

§2.3. Pullbacks of Lie groupoids seem to have first been considered by Pradines [1986b] and in [Mackenzie, 1987c]. For set groupoids, pullbacks always exist, of course. In that category they provide a concept dual to the universal morphisms of Higgins [1971].

The need for the additional condition in 2.3.1 was pointed out by Pradines [1989].

§2.4. Most of this section has been taken directly from [Higgins and Mackenzie, 1990b]. The terminology *weak pullback* from [Higgins and Mackenzie, 1990b] has been replaced by *versal* as used by Pradines [1986a]. Further, I have replaced the uses of *regular* in [Higgins and Mackenzie, 1990b] by *uniform* (see below).

For the notion of fibration of set groupoids see [Brown, 1970], where it is treated explicitly as an algebraic model of the path–lifting property for fibrations of topological spaces.

Pradines [1986b] introduced a notion of *extenseurs réguliers* and gave a first isomorphism theorem for them. These results were then recovered in [Higgins and Mackenzie, 1990b, §4] as a special case of the treatment given here.

A *congruence* (2.4.5) on a Lie groupoid $G \rightrightarrows M$ is equivalent to a Lie double subgroupoid of the double Lie groupoid $(G \times G; G, M \times M; M)$; see [Higgins and Mackenzie, 1990b, p.105].

Interesting examples of fibrations arise in the theory of double Lie groupoids. In [Brown and Mackenzie, 1992], for example, a double version of local triviality is defined using a fibration condition, and it is shown that double groupoids for which this fibration is split can be described by an explicit construction in terms of crossed modules. In [Mackenzie, 1992, §3] it is noted that (under a weak extra condition) the maps from a double groupoid to its component groupoids are fibrations; this is useful in the description of affinoids (see also [Mackenzie, 2000a]).

Proposition 2.4.14 is from [Brown and Mackenzie, 1992].

§2.5. The general notion of groupoid action and the *general semidirect products* of this section were introduced for set groupoids by Brown [1970, 1972], following work by Frölich; the treatment given here follows [Higgins and Mackenzie, 1990a, §3]. General semidirect products have usually been denoted by the standard semidirect product symbol $\ltimes$; I have introduced $\sqsubset\!\sqsubset$ here to keep the distinctions clear.

A still more general notion of action of one groupoid on another is used by Xu [1992].

The notion of PBG–groupoid comes from [Mackenzie, 1988b] and [Mackenzie, 1987b]. This concept is essentially of double nature: a PBG–groupoid is a principal G–bundle object in the category of locally trivial Lie groupoids. The terminology is meant to suggest that Υ is not merely a G–groupoid (a groupoid with a group action from G) but a principal bundle with respect to that action.

§2.6. The general scheme followed in this section is from [Pradines, 1986b], but the account here (which broadly follows [Higgins and Mackenzie, 1990b]) varies many details. The *extenseurs réguliers* of [Pradines, 1986b] are here called *uniform fibrations*, since the meaning given to 'regular' in [Higgins and Mackenzie, 1990b] does not correspond to Pradines' 'régulier'. The *inducteurs surmersifs* of [Pradines, 1986b] are here called inductors. Note that in [Pradines, 1986b], the notation f_0^*G refers to what is here denoted $f^{\Downarrow}G$.

There is an interesting analysis of general morphisms of set groupoids in [Higgins, 1971]; the results summarized in §2.6 may be regarded as the smooth version of (a part of) this.

Occasions arise on which it would be useful to have versions of the results of §2.4 and §2.5 without the assumption that the base–maps are surjective submersions. In that generality the framework which is used here is not available (because the pullback groupoids generally do not exist) and it is generally necessary to proceed on an *ad hoc* basis.

Notions of equivalence. An inductor $F\colon G' \to G$, $f\colon M' \to M$, with the further property that the map $\overline{\beta}\colon f^{!}G \to M$, $(g, m') \mapsto \beta g$, is a surjective submersion, is generally known as a *weak equivalence*. Inverting the weak equivalences gives the notion of Morita equivalence for Lie groupoids. See [Moerdijk and Mrčun, 2003, §5.4] for a clear recent account. Morita equivalence is fundamental in several areas where groupoids are applied: in C^*–algebra theory [Muhly et al., 1987], in aspects of symplectic and Poisson geometry [Xu, 1991a,b], [Landsman, 2002], and in foliation theory [Moerdijk and Mrčun, 2003].