# DOUBLE LIE ALGEBROIDS AND SECOND–ORDER GEOMETRY, I

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The original preprint was written in plainTEX in 1990, with diagrams partly hand drawn. It was put into LATEX in December 1997 but is not guaranteed free of errors arising from the conversion. The text has not been changed in any way, other than the following:

- several diagrams are now numbered as "Figures", and references to the Figure numbers have been introduced into the text;
- in diagrams of morphisms of double structures, some identity arrows have been omitted for clarity;
- *Advances* made many small stylistic changes which are not reproduced here.

This is the first part of a two-part paper, the purpose of which is to show that the algebra of double groupoids—the algebra of squares, to paraphrase [BrH]—underlies a possible second-order geometry in the same way that parallel translation or path lifting—the algebra of paths—underlies first-order geometry. Here we are regarding the basic object of interest in geometry as, for example, a metric or, more generally, a $G$-structure or, more generally still, an abstract principal bundle. With these terms as basic, the standard concept of infinitesimal connection is a first-order concept. It was shown in [M1], following work of Pradines, that standard connection theory may be deduced from the Lie theory of (locally trivial) Lie groupoids and (transitive) Lie algebroids: Lie groupoids are a generalization of principal bundles which permit an analogy with Lie groups, and imitating the construction of the Lie algebra of a Lie group yields a first-order invariant, the *Lie algebroid*, which may be identified, for a Lie groupoid corresponding to a principal bundle, with the Atiyah sequence of the bundle. One can then show, for example, that the correspondence between Lie subgroupoids and Lie subalgebroids (with suitable connectivity and transitivity assumptions) includes the Ambrose-Singer and Reduction theorems of connection theory, and that the resolution of the integrability problem for transitive Lie algebroids gives criteria for the existence of principal bundle connections with prescribed curvature form. (It should be mentioned that these criteria are uninteresting when the structure group is semisimple and, in particular, give no new information in the Riemannian case.) See [M1] for a full account of these matters and further references.

In this paper we propose to develop a corresponding Lie theory for double Lie groupoids. Ordinary groupoids may be regarded as consisting of elements of length—groupoid elements are typically pictured as arrows from one point to another of a specific base manifold, as, for example, the Lie groupoid of linear isomorphisms between the fibres of a vector bundle. In a similar way the elements of a double groupoid can be regarded as elements of area, or more precisely as squares, the four sides of which give the sources and targets of two distinct ordinary groupoid structures. Whereas the Lie algebroid of an ordinary groupoid is a first-order invariant, and in the case of the groupoid whose elements are pairs of points from a manifold $B$, gives the ordinary tangent bundle $TB$, the double Lie algebroid of a double Lie groupoid is a second-order invariant in an essential way, and for a manifold $B$ and the double groupoid of quadruples of points from $B$ (2.4), gives the second-order tangent bundle $T^2B = T(TB)$.

Being a second-order invariant, the double Lie algebroid of a double Lie groupoid requires two steps for its construction. A double Lie groupoid consists of a manifold $S$ with two distinct Lie groupoid structures on (generally distinct) base manifolds $H$ and $V$, which are themselves Lie groupoids on a common base manifold $B$, the compatibility condition between the two structures being that the structure maps (source, target, composition and identity) of each structure on $S$ are morphisms with respect to the other (see the discussion following 2.1). The groupoid structure on $S$ with base $V$ is the *horizontal structure*; that with base $H$ is the *vertical structure*; $H$ and $V$ are the *side groupoids*, and $B$ is the *double base*. The double Lie algebroid of $S$ is obtained by first applying the ordinary Lie functor to (say) the vertical structure on $S$; this produces a Lie algebroid which inherits, from the horizontal structure of $S$, a Lie groupoid structure with base the Lie algebroid of $V$. A structure of this type, which is a groupoid object in the category of Lie algebroids, we call an $\mathcal{LA}$-groupoid (§4). Applying the Lie functor again, to the groupoid structure on this $\mathcal{LA}$-groupoid, gives the double Lie algebroid of $S$. (In fact this gives a double vector bundle with only one Lie algebroid structure; in order to obtain the other it is necessary to consider the above process in the reverse order, applying the Lie functor first horizontally and then vertically, and to identify the result with the first double vector bundle. Full details will be given in the second part of the paper.)

In this first part we are mainly concerned with the first of these steps, and with the general theory of $\mathcal{LA}$-groupoids. There are two main strands to the theory as we have developed it here. Ordinary connection theory takes place in locally trivial Lie groupoids or locally trivial principal bundles; indeed an infinitesimal connection can be regarded as an attempt to give, on the level of the tangent bundle, a product decomposition which exists only locally in the bundle or groupoid itself. The first strand of this paper is, then, concerned with local triviality for double groupoids, as introduced in [BrM], and the corresponding concepts for their $\mathcal{LA}$-groupoids and double Lie algebroids. For ordinary Lie groupoids or principal bundles, local triviality, as usually defined, expresses the structure locally in terms of a Lie group and a contractible space. Alternatively, a Lie groupoid $G$ on base $B$ is locally trivial if its anchor map $G \to B \times B,\ g \mapsto (\beta g, \alpha g)$, which sends each arrow to its two end points, is a surjective submersion. In [BrM] a double Lie groupoid was analogously defined to be *locally trivial* if both of its anchor maps are not only surjective submersions, but $s$-fibrations, a stronger condition which takes account of the double structure. It then followed that a locally trivial double Lie groupoid $S$ is determined by a simple diagram of ordinary Lie groupoids—the *core diagram*—consisting of a *core groupoid* $K$, whose base is the double base of $S$, and two surjective submersive morphisms $\partial_H\colon K \to H$ and $\partial_V\colon K \to V$ to the side groupoids of $S$, whose kernels commute in $K$. In §5 here we establish similar results for $\mathcal{LA}$-groupoids and begin the development of a connection theory for double Lie groupoids by showing (5.10) that connections in an $\mathcal{LA}$-groupoid which respect the groupoid structure are determined by vector bundle splittings of the core diagram sequences. In the second part of the paper we will introduce a notion of "double connection" in a double Lie algebroid, and show that for the double Lie algebroids of certain double groupoids constructed from ordinary groupoids, these double connections induce second-order connections in the ordinary groupoids; that is, laws of liftings for 2-jets. The square-lifting associated with these double connections will thereby give a geometric interpretation for second-order connections.

Secondly, double groupoid structures have arisen in connection with Poisson Lie groups. A compatible Poisson bracket $\{\ ,\}$ on a Lie group $G$ induces a Lie algebra structure on the vector space dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ of $G$ in such a way that the vector space sum $\mathcal{G} \oplus \mathcal{G}^*$ acquires a Lie algebra structure with $\mathcal{G}$ and $\mathcal{G}^*$ as subalgebras and the remaining terms of the bracket given by the coadjoint actions of $\mathcal{G}$ and $\mathcal{G}^*$ on each other; this is the *Lie bigebra* of the Poisson Lie group $(G, \{\ ,\})$ ([Dr], [LW1]; see also [K-SM] and [Mj]). Under certain conditions a construction in the opposite direction is also possible. If $G^*$ is the simply-connected Lie group corresponding to $\mathcal{G}^*$ (so that $G^*$ is the dual group for $G$), and if the coadjoint actions of $G$ on $\mathcal{G}^*$ and of $G^*$ on $\mathcal{G}$ can be suitably integrated, then the product manifold $G \times G^*$ acquires a Lie group structure with $G$ and $G^*$ as Lie subgroups and Lie algebra $\mathcal{G} \oplus \mathcal{G}^*$. ([LW1])

Abstracting the structure of $\mathcal{G} \oplus \mathcal{G}^*$ and $G \times G^*$ leads to concepts of *double Lie algebra* and *double Lie group*: a double Lie group is a Lie group $S$ together with Lie subgroups $H$ and $V$ such that the multiplication map $V \times H \to S$ is a diffeomorphism, and similarly a double Lie algebra is a Lie algebra $\mathcal{S}$ together with Lie subalgebras $\mathcal{H}$ and $\mathcal{V}$ such that addition $\mathcal{V} \oplus \mathcal{H} \to \mathcal{S}$ is a bijection. There is a Lie theory for double Lie groups and double Lie algebras which has all the properties one would wish. ([LW1])

Double Lie groups in this sense at first appear unrelated to double Lie groupoids as defined in 2.1. Nonetheless, a double Lie group may be considered to be a double Lie groupoid in a natural, and reasonably evident, way; the actions of $H$ on $V$ and of $V$ on $H$ define groupoid structures on $S$ with bases $V$ and $H$ respectively, which give a double Lie groupoid in the sense of 2.1, but of a very special type. Namely the elements of $S$, viewed as squares, are determined by any two nonparallel sides. We call double groupoids satisfying this condition *vacant* (2.11) and show (2.12, 2.13) that vacant double groupoids always arise from double Lie groups in the sense of [LW1], or from the natural extension of that concept to groupoids.

The $\mathcal{LA}$-groupoids associated to a vacant double groupoid satisfy a similar vacancy condition. We show (4.12) that if $(G, \{\ ,\})$ is a Poisson Lie group, then the cotangent bundle $T^*G$ has a natural structure of a vacant $\mathcal{LA}$-groupoid; if the Poisson structure of $G$ arises from a double Lie group $G \times G^*$ then one $\mathcal{LA}$-groupoid of $G \times G^*$, considered as a double Lie groupoid, is precisely $T^*G$. (On the other hand, $T^*G$, for certain Poisson Lie groups $G$, may integrate to a double Lie groupoid which is not vacant ([LW2]).) Similarly the double Lie algebroid associated to $T^*G$ is $\mathcal{G} \oplus \mathcal{G}^*$, the Lie bigebra of $(G, \{\ ,\})$, with two Lie algebroid structures that will be detailed elsewhere.

This example shows, in particular, that $\mathcal{LA}$-groupoids (whose double base is a point) are intermediate between double Lie algebras and double Lie groups in the same way that Poisson Lie groups are intermediate between Lie bigebras and double Lie groups of the form $G \times G^*$.

It is characteristic of vacant double groupoids that each of the side groupoids acts upon the other. In the same way a vacant $\mathcal{LA}$-groupoid with side Lie algebroid $A$ and side groupoid $G$ defines actions of $G$ on $A$ and of $A$ on $G$. In the case of the cotangent bundle of a Poisson Lie group $G$ these are, respectively, the coadjoint action of $G$ on $\mathcal{G}^*$ and the dressing action of $\mathcal{G}^*$ on $G$. The general treatment given here shows that the equations characteristic of dressing transformations are precisely the compatibility conditions for $\mathcal{LA}$-groupoids (4.12).

The class of vacant double Lie groupoids also includes a class of structures variously known as *affinoid structures* (Weinstein [W2]), or *pregroupoids* (Kock [K]), and which have been discovered intermittently since Baer (see references in [K] and [W2]). We observe that these are also vacant double Lie groupoids and give simple proofs of the equivalence of this notion with that of "principal bundle with structure Lie groupoid" and with the "butterfly diagrams" of Pradines [P2] (a type of Morita equivalence). This material is included because the associated Lie algebroid constructions will be a useful test case in the second part of the paper; and because we believe our proof of 3.5, which takes full advantage of the double groupoid structure, is particularly simple.

The organization of this first part is as follows. In §1 we give background material on double vector bundles and their cores; this will be needed in the second part of the paper, but is also intended as an introduction to the core structure of double groupoids, which is recalled in the first part of §2. The second part of §2, from 2.9 on, describes the double groupoid structure of a double Lie group, and shows that all vacant double groupoids arise from situations of this type. In particular we show that the compatibilty conditions between the two actions in a double Lie group arise in a simple and natural diagrammatic way if the group elements are pictured as squares. §3 treats the case of vacant double Lie groupoids whose side groupoids are equivalence relations—the affinoids [W2], or pregroupoids [K]—and shows that they are equivalent to principle bundles with structure Lie groupoid, and to the butterfly diagrams of Pradines [P2]. §4 introduces $\mathcal{LA}$-groupoids, calculating the $\mathcal{LA}$-groupoids of basic examples of double Lie groupoids, and showing that vacant $\mathcal{LA}$-groupoids are determined by a pair of actions which satisfy differentiated forms of those required for a vacant double Lie groupoid. §5 defines the core diagram of suitable $\mathcal{LA}$-groupoids and shows that in the presence of appropiate transitivity conditions, an $\mathcal{LA}$-groupoid can be reconstructed from its core diagram. From this it follows that suitable connections in a transitive $\mathcal{LA}$-groupoid are determined by their restriction to the core.

This paper makes extensive use of the results and conceptual framework of [HM1] and [BrM], and our notation and conventions largely follow these two papers, as well as [M1]. However, in [M1] and elsewhere we have used the term "Lie groupoid" to refer to a differentiable groupoid satisfying the local triviality condition described above. Here (and henceforth) by a Lie groupoid we will mean what we have previously called a differentiable groupoid, and when local triviality conditions are used, they will be explicitly stated. This change of usage seems consonant with the growing prominence of symplectic groupoids ([W1]). Some further smaller changes are noted at the starts of §2 and §4. All manifolds are $C^\infty$, real, Hausdorff and second-countable.

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## 1 PRELIMINARIES ON DOUBLE VECTOR BUNDLES

This section is a brief account of some basic constructions with double vector bundles. This material will be needed in the second part of the paper, but will also serve here as background to §2 and §5. Double vector bundles are particular cases of both double Lie groupoids and $\mathcal{LA}$-groupoids, and so some of this material is in principle covered in [BrM] or §5; historically, however, the double vector bundle case came first, and also admits features not available for general double Lie groupoids or $\mathcal{LA}$-groupoids. A very full account of what can be done with double vector bundles is given by Pradines [P1]; see also [B].

**Definition 1.1.** *A* double vector bundle $(E; E^H, E^V; B)$ *is a system of four vector bundle* *structures*

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{q}_H} & E^V \\ \widetilde{q}_V \Big\downarrow & & \Big\downarrow q_V \\ E^H & \xrightarrow[q_H]{} & B \end{array} \qquad (1)$$

*in which $E$ has two vector bundle structures, on bases $E^V$ and $E^H$, which are themselves vector bundles on $B$, such that each of the four structural maps of each vector bundle structure on $E$ (namely the bundle projection, addition, scalar multiplication and the zero section) is a morphism of vector bundles with respect to the other structure.* ■

We refer to $E^H$ and $E^V$ as the *side bundles* of $E$. The addition, scalar multiplication and subtraction in $E^V$ and $E^H$ are denoted by the usual symbols $+$, juxtaposition, and $-$; we usually denote elements of $E^H$ by script letters $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, ..., and elements of $E^V$ by upper-case roman letters X, Y, Z, ..., so that the context will make clear which bundle is relevant. We distinguish the two zero-sections, however, writing $0^H: B \to E^H, b \mapsto 0^H_b$ and $0^V: B \to E^V, b \mapsto 0^V_b$. In the *vertical bundle structure* on $E$ with base $E^H$ we write $\underset{V}{+}$ , $\underset{V}{\cdot}$ , $\underset{V}{-}$ , and $\widetilde{0}^V: E^H \to E$, $\mathcal{X} \mapsto \widetilde{0}^V_{\mathcal{X}}$ for the zero-section. Similarly, in the *horizontal bundle structure* on $E$ with base $E^V$ we write $\underset{H}{+}$ , $\underset{H}{\cdot}$ , $\underset{H}{-}$ and $\widetilde{0}^H: E^V \to E$, $X \mapsto \widetilde{0}^H_X$. For $b \in B$ the *double zero* $\widetilde{0}^H_{0^V_b} = \widetilde{0}^V_{0^H_b}$ is denoted $\odot_b$. The two structures on $E$, namely $(E, \widetilde{q}_H, E^V)$ and $(E, \widetilde{q}_V, E^H)$, will be denoted $\widetilde{E}^H$ and $\widetilde{E}^V$, respectively.

The compatibility conditions on the four structures can now be written more explicitly as follows. The bundle projection $\widetilde{q}_H: \widetilde{E}^V \to E^V$ must be a vector bundle projection over $q_H: E^H \to B$, as must the zero-section $\widetilde{0}^H: E^V \to \widetilde{E}^V$ over $0^H: B \to E^H$. If one takes the pullback

$$\begin{array}{ccc} \widetilde{E}^V \underset{E^V}{\oplus} \widetilde{E}^V & \longrightarrow & \widetilde{E}^V \\ \Big\downarrow & & \Big\downarrow \widetilde{q}_H \\ \widetilde{E}^V & \xrightarrow[\widetilde{q}_H]{} & E^V \end{array}$$

in the category of vector bundles, the addition $\underset{H}{+} : \widetilde{E}^V \underset{E^V}{\oplus} \widetilde{E}^V \to \widetilde{E}^V$ must be a morphism of vector bundles over $+: E^H \underset{B}{\oplus} E^H \to E^H$. Lastly, taking the pullback

$$\begin{array}{ccc} (E^V \times \mathbf{R}) \underset{E^V}{\oplus} \widetilde{E}^V & \longrightarrow & \widetilde{E}^V \\ \Big\downarrow & & \Big\downarrow \widetilde{q}_H \\ E^V \times \mathbf{R} & \xrightarrow[q]{} & E^V \end{array}$$

where $E^V \times \mathbf{R}$ is regarded as a vector bundle over $B \times \mathbf{R}$ in the obvious way, and $q$ is the canonical projection, regarded as a morphism of vector bundles over $B \times \mathbf{R} \to B$, the scalar multiplication $(E^V \times \mathbf{R}) \underset{E^V}{\oplus} \widetilde{E}^V \to \widetilde{E}^V$ must be a morphism of vector bundles over the scalar multiplication $(B \times \mathbf{R}) \underset{B}{\oplus} E^H \to E^H$. Once these four conditions are satisfied, the corresponding conditions in the vertical structure follow.

In particular,

$$(\xi \underset{H}{+} \eta) \underset{V}{+} (\zeta \underset{H}{+} \omega) = (\xi \underset{V}{+} \zeta) \underset{H}{+} (\eta \underset{V}{+} \omega) \tag{2}$$

for quadruples $\xi, \eta, \zeta, \omega \in E$ such that $\widetilde{q}_H(\xi) = \widetilde{q}_H(\eta)$, $\widetilde{q}_H(\zeta) = \widetilde{q}_H(\omega)$, $\widetilde{q}_V(\xi) = \widetilde{q}_V(\zeta)$, and $\widetilde{q}_V(\eta) = \widetilde{q}_V(\omega)$. Also,

$$t \underset{V}{\cdot} (\xi \underset{H}{+} \eta) = t \underset{V}{\cdot} \xi \underset{H}{+} t \underset{V}{\cdot} \eta, \qquad t \underset{V}{\cdot} (u \underset{H}{\cdot} \xi) = u \underset{H}{\cdot} (t \underset{V}{\cdot} \xi)$$

for $t,\ u \in \mathbf{R}$ and $\xi, \eta \in E$ with $\widetilde{q}_H(\xi) = \widetilde{q}_H(\eta)$.

Given a double vector bundle $(E; E^H, E^V; B)$, interchanging the horizontal and vertical structures gives another double vector bundle $(E; E^V, E^H; B)$, which we call the *flip* of $E$. A *morphism of double vector bundles* $(\varphi; \varphi_H, \varphi_V; f){:}\, (E; E^H, E^V; B) \to (D; D^H, D^V; B')$ consists of maps $\varphi{:}\, E \to D$, $\varphi_H{:}\, E^H \to D^H$, $\varphi_V{:}\, E^V \to D^V$, $f{:}\, B \to B'$ such that each of $(\varphi, \varphi_H)$, $(\varphi, \varphi_V)$, $(\varphi_H, f)$ and $(\varphi_V, f)$ is a morphism of the relevant vector bundles.

Until 1.3, consider a fixed double vector bundle $(E; E^H, E^V; B)$. Denote by $E^C$ the intersection of the kernels of the two bundle projections;

$$E^C = \{\xi \in E \mid \exists b \in B{:}\, \widetilde{q}_H(\xi) = 0^V_b,\ \widetilde{q}_V(\xi) = 0^H_b\}.$$

This is an embedded submanifold of $E$, and inherits a well-defined vector bundle structure with base $B$, projection $q_C = q_V \circ \widetilde{q}_H = q_H \circ \widetilde{q}_V$ and addition and scalar multiplication the restrictions of either of the operations on $E$. That the two additions coincide on $E^C$ follows from $\xi \underset{H}{+} \eta = (\xi \underset{V}{+} \widetilde{0}_b) \underset{H}{+} (\widetilde{0}_b \underset{V}{+} \eta) = (\xi \underset{H}{+} \widetilde{0}_b) \underset{V}{+} (\widetilde{0}_b \underset{H}{+} \eta) = \xi \underset{V}{+} \eta$, using (2). From this is follows that $t \underset{H}{\cdot} \xi = t \underset{V}{\cdot} \xi$ for integers $t$, and consequently for rational $t$, and thence for all real $t$ by continuity. It is now easy to prove that $(E^C, q_C, B)$ is a (smooth) vector bundle, which, following Pradines ([P1, C§2]), we call the *core* of $(E; E^H, E^V; B)$.

**Proposition 1.2.** *([P1, C§3]) For a double vector bundle $(E; E^H, E^V; B)$ there is an exact sequence*

$$q_H^* E^C \rightarrowtail \widetilde{E}^V \xrightarrow{\widetilde{q}_H^*} \!\!\!\!\twoheadrightarrow q_H^* E^V \tag{3}$$

*of vector bundles over $E^H$, and an exact sequence*

$$q_V^* E^C \rightarrowtail \widetilde{E}^H \xrightarrow{\widetilde{q}_V^*} \!\!\!\!\twoheadrightarrow q_V^* E^H \tag{4}$$

*of vector bundles over $E^V$. Here the injections are $(\mathcal{X}, \xi) \mapsto \widetilde{0}^V_{\mathcal{X}} \underset{H}{+} \xi$ and $(X, \xi) \mapsto \widetilde{0}^H_X \underset{V}{+} \xi$, respectively, and $\widetilde{q}_H^*$ and $\widetilde{q}_V^*$ denote the maps induced by $\widetilde{q}_H$ and $\widetilde{q}_V$ into the pullback bundles.*

PROOF: Take $\mathcal{X} \in E^H_b$, $\xi \in E^C_b$ where $b \in B$. Then both $\widetilde{0}^V_{\mathcal{X}}$ and $\xi$ project under $\widetilde{q}_H$ to $0^V_b$. So $\widetilde{0}^V_{\mathcal{X}} \underset{H}{+} \xi$ is defined and also projects under $\widetilde{q}_H$ to $0^V_b$. Conversely, suppose that $\zeta \in E$ has $\widetilde{q}_H(\zeta) = 0^V_b$ for some $b \in B$. Write $\mathcal{X} = \widetilde{q}_V(\zeta)$. Then $\zeta \underset{H}{-} \widetilde{0}^V_{\mathcal{X}}$ is defined and $\widetilde{q}_H(\zeta \underset{H}{-} \widetilde{0}^V_{\mathcal{X}}) = 0^V_b$. On the other hand, $\widetilde{q}^V(\zeta \underset{H}{-} \widetilde{0}^V_{\mathcal{X}}) = \mathcal{X} - \mathcal{X} = 0^H_b$. So $\zeta \underset{H}{-} \widetilde{0}^V_{\mathcal{X}} \in E^C_b$. This establishes the exactness of (3). The proof of (4) is similar. ∎

We refer to (3) and (4) as the *core sequences* of $(E; E^H, E^V; B)$.

**Example 1.3** Let $E^H, E^V$ and $E^C$ be any three vector bundles on the one base $B$, and write $E$ for the direct sum $E^H \oplus E^V \oplus E^C$ over $B$. Then $E$ may be regarded as the direct sum $q_H^* E^V \oplus q_H^* E^C$ over $E^H$, and as the direct sum $q_V^* E^H \oplus q_V^* E^C$ over $E^V$, and with respect to these two structures, $E$ is a double vector bundle with side bundles $E^H$ and $E^V$ and core bundle $E^C$. Note that the two core sequences are canonically split. We call this the *trivial double vector bundle over $E^H$ and $E^V$ with core $E^C$*. In [P1] it is a “fibré vectoriel double decomposé”. ■

**Example 1.4** ([P1], [B]) Consider a vector bundle $q: E \to B$. Then $TE$ has its standard structure as the tangent bundle $(TE, p_E, E)$ over $E$, but is also a vector bundle over $TB$ with projection $T(q): TE \to TB$ and addition and scalar multiplication defined as the tangents of the addition and scalar multiplication in $E$. With these two structures $TE$ is a double vector bundle

$$\begin{array}{ccc} TE & \xrightarrow{\;p_E\;} & E \\ {\scriptstyle T(q)}\downarrow & & \downarrow{\scriptstyle q} \\ TB & \xrightarrow[\;p_B\;]{} & B. \end{array}$$

We will want to use this example equally often with the horizontal and vertical structures reversed, and so we adopt the following notation: In $(TE, p_E, E)$ we use $+$, juxtaposition, $-$ and $\widetilde{0}: E \to TE$, $e \mapsto \widetilde{0}_e$ for the vector bundle operations. In $(TE, T(q), TB)$ we write $\mathbin{+\!\!\!+}$, $\bullet$, $\boldsymbol{-}$, and $T(0): TB \to TE$, $X \mapsto T(0)(X)$. (The notation $\mathbin{+\!\!\!+}$ is from [B].) By $T(E)_e$ we will always mean the fibre over $e \in E$ of the tangent bundle projection $p_E$.

To find the core of $TE$, notice first that the kernel of $T(q): TE \to TB$ is the vertical subbundle $T^q E$. Since $T^q(E)_e = T(E_{q(e)})_e \cong \{e\} \times E_{q(e)}$, this subbundle is naturally isomorphic to the pullback $q^* E = E \underset{B}{\times} E$, with $(e_1, e_2) \in E_b \times E_b$ representing the tangent vector to $E_b$ whose tail is at $e_1$ and which is parallel to $e_2$. Now $p_E: TE \to E$ restricted to $q^* E$ is $(e_1, e_2) \mapsto e_1$ and so the core of $TE$ can be naturally identified with $E$ itself under $e \mapsto (0_{q(e)}, e)$. The natural injection $p_B^* E \to TE$ maps $(X, e) \in T(B)_b \times E_b$ to $(0_b, e) + T(0)(X) \in T(E)_{0_b}$, and is an isomorphism onto the kernel of $p_E: TE \to E$. ■

Taking $E = TB$ in this example we obtain the *double tangent bundle*

$$\begin{array}{ccc} T^2B & \xrightarrow{\;p_{TB}\;} & TB \\ {\scriptstyle T(p_B)}\downarrow & & \downarrow{\scriptstyle p_B} \\ TB & \xrightarrow[\;p_B\;]{} & B. \end{array}$$

We will write $p$ for $p_B$ and $p_T$ for $p_{TB}$ if no confusion seems likely. This example has the important extra property that there is a *canonical involution* $J: T^2B \to T^2B$ which is an isomorphism of double vector bundles from $T^2B$ to its flip, as in Figure 1, leaving the side

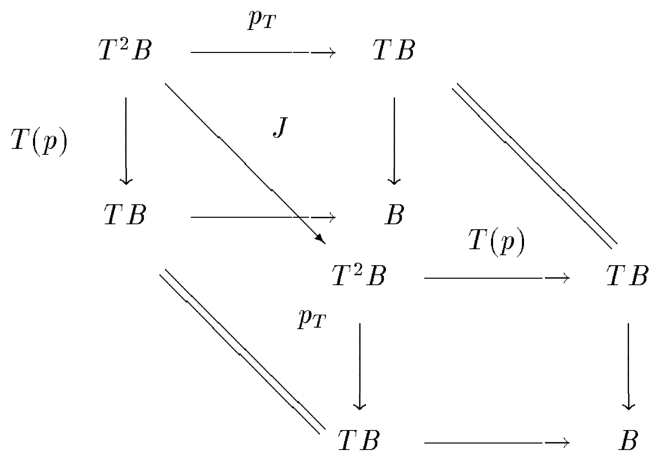



Figure 1.

bundles fixed. Thus $p_T \circ J = T(p)$ and $T(p) \circ J = p_T$. Further, $J$ is natural in the sense that if $f: B \to B'$ is any smooth map then $T^2(f) \circ J = J' \circ T^2(f)$. For further details see, for example, [B], [P1].

Given any morphism of double vector bundles

$$(\varphi; \varphi_H, \varphi_V; f): (E; E^H, E^V; B) \to (D; D^H, D^V; B')$$

satisfying suitable locally constant rank conditions, there will exist kernels $K^H \subseteq E^H$ of $(\varphi_H, f)$ and $\widetilde{K}^H \subseteq E$ of $(\varphi, \varphi_V)$. These form a double vector subbundle (in an obvious sense) as in Figure 2. In particular, if $\varphi: E \to E'$, $f: B \to B'$ is a morphism of vector bundles, then

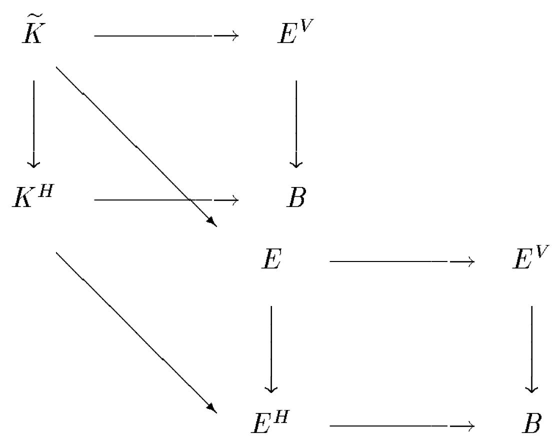



Figure 2.

there is a double vector subbundle

$$\begin{array}{ccc} T^{\varphi}E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ T^{f}B & \longrightarrow & B \end{array} \qquad \text{of} \qquad \begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ T(q)\big\downarrow & & \big\downarrow q \\ TB & \xrightarrow[p_B]{} & B \end{array}$$

whose core is the kernel (in the ordinary sense) of $(\varphi, f)$. In the case where $E = TB,\ E' = TB',\ \varphi = T(f)$ this is a double vector subbundle

$$\begin{array}{ccc} T^{T(f)}TB & \xrightarrow{p_T} & TB \\ T(p)\big\downarrow & & \big\downarrow p \\ T^{f}B & \xrightarrow[p]{} & B \end{array} \tag{5}$$

of $T^2B$. The following simple result will be crucial in the second part of the paper.

**Proposition 1.5.** *The canonical involution carries (5) to*

$$\begin{array}{ccc} T(T^{f}B) & \xrightarrow{T(p)} & TB \\ p_T\big\downarrow & & \big\downarrow p \\ T^{f}B & \xrightarrow[p]{} & B, \end{array}$$

*the double vector bundle $(TE; E, TB; B)$ for $E = T^f B$.*

PROOF: This follows from the naturality of $J$ and a dimension-counting argument. ∎

The two core sequences associated with $T(T^f B)$ are

$$T^f B \underset{B}{\times} T^f B \rightarrowtail T(T^f B) \xrightarrow{T(p)^*} \!\!\!\!\twoheadrightarrow T^f B \underset{B}{\times} TB,$$

an exact sequence of vector bundles over $T^f B$, and

$$TB \underset{B}{\times} T^f B \rightarrowtail T(T^f B) \xrightarrow{p_T^*} \!\!\!\!\twoheadrightarrow TB \underset{B}{\times} T^f B,$$

an exact sequence of vector bundles over $TB$.

# 2 DOUBLE LIE GROUPOIDS AND DOUBLE LIE GROUPS

As noted in the introduction, we use the term "Lie groupoid" for what in [M1] and elsewhere we have called a differentiable groupoid. We thus no longer assume that Lie groupoids are locally trivial, and when local triviality conditions are used, they will be stated explicitly. The first part of this section sets up notation and recalls results from [BrM] on the structure of locally trivial double Lie groupoids; in 2.9 we begin the analysis of the double Lie groupoid structure associated with a double Lie group.

We begin by recalling the main classes of morphisms of (ordinary) Lie groupoids; see [P3], [HM1] and [HM2] and the references given there.

We will use the notation $G \rightrightarrows B$ to indicate briefly that $G$ is a groupoid on base $B$. Here the two arrows should be thought of as the source $\alpha: G \to B$ and target $\beta: G \to B$ maps. The anchor map $G \to B \times B,\ g \mapsto (\beta g, \alpha g)$ we generically denote by $\chi$. The identity element corresponding to $b \in B$ we denote by $1_b$, not, as in [M1], by $\tilde{b}$. The domain of the multiplication map we denote by $G * G = \{(g_2, g_1) \in G \times G \mid \alpha(g_2) = \beta(g_1)\}$, and the multiplication map itself by $\kappa$. Thus a groupoid product $g_2 g_1 = \kappa(g_2, g_1)$ is defined if and only if $\alpha(g_2) = \beta(g_1)$. Lastly, the *division map* of $G$ we take to be $\delta: G \underset{\alpha}{\times} G \to G,\ (g', g) \mapsto g' g^{-1}$, where $G \underset{\alpha}{\times} G$ denotes the pullback of $\alpha$ over itself.

Consider a morphism of Lie groupoids $\varphi: G' \longrightarrow G,\ f: B' \longrightarrow B$, and form the pullback manifold

$$\begin{array}{ccc} f^*G & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \alpha \\ B' & \underset{f}{\longrightarrow} & B. \end{array}$$

Let $\varphi^*: G' \to f^*G$ be the induced map $g' \mapsto (\alpha' g', \varphi(g'))$. Then $(\varphi, f)$ is a *fibration* if $\varphi^*$ is a surjective submersion, and $(\varphi, f)$ is an *action morphism* if $\varphi^*$ is a diffeomorphism. If $f: B' \to B$ is also a surjective submersion then we speak of an *s-fibration* or an *s-action morphism*. (In [HM1] and [HM2], the definition of fibration included the condition on the base map.)

When $f$ is a surjective submersion one can also form the pullback (which is in fact the pullback groupoid),

$$\begin{array}{ccc} f^{**}G & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \chi \\ B' \times B' & \underset{f \times f}{\longrightarrow} & B \times B. \end{array}$$

There is now an induced map $\varphi^{**}: G' \to f^{**}G,\ g' \mapsto (\beta' g', \varphi(g'), \alpha' g')$. In this case, $(\varphi, f)$ is a *regular fibration* if $\varphi^{**}$ is a surjective submersion, and is an *inductor* if $\varphi^{**}$ is a diffeomorphism.

These four classes of morphism correspond to basic algebraic constructions such as action, quotient and pullback. We briefly recall the details we will use most often here.

A (left) action of a Lie groupoid $G \rightrightarrows B$ on a smooth map $f: B' \to B$ is a map $G \underset{B}{\times} B' \to B'$, $(g, b') \mapsto gb'$, where $G \underset{B}{\times} B' = \{(g, b') | \ \alpha(g) = f(b')\}$, such that (i) $f(gb') = \beta(g)$; (ii) $g_2(g_1 b') = (g_2 g_1)b'$; (iii) $1_{f(b')}b' = b'$ for all $g, g_2, g_1 \in G$ and $b' \in B'$ which are suitably compatible. Given such an action one constructs the *action groupoid*, or *transformation groupoid*, $G \ltimes f$ or $G \ltimes B'$, on the manifold $G \underset{B}{\times} B'$, with base $B'$, by defining $\alpha'(g, b') = b'$, $\beta'(g, b') = gb'$, and $(g_2, g_1 b')(g_1, b') = (g_2 g_1, b')$. Then $G \ltimes f \to G$, $(g, b') \mapsto g$ is a morphism of Lie groupoids over $f$ and an action morphism. Conversely, given an action morphism $\varphi: G' \to G$, $f: B' \to B$, the formula $gb' = \beta'(\varphi^{*-1}(b', g))$ defines an action of $G$ on $f$. This correspondence between actions and action morphisms is bijective. (See [M1, II 4.20 *et seq.*] and references given there.)

A right action of $G \rightrightarrows B$ on $f$ is similarly defined as a map $B' \underset{B}{\times} G \to B'$, $(b', g) \mapsto b'g$, where now $B' \underset{B}{\times} G$ is the pullback of $f$ and $\beta$, such that (i) $f(b'g) = \alpha(g)$; (ii) $(b'g_1)g_2 = b'(g_1 g_2)$; (iii) $b'1_{f(b')} = b'$ for all compatible $g, g_2, g_1 \in G$, $b' \in B'$. If $(g, b') \mapsto gb'$ is a left action then the usual formula $b'g = g^{-1}b'$ defines a right action. Given a right action, we take the *right action groupoid* $f \rtimes G$ to be $B' \underset{B}{\times} G$ with $\alpha'(b', g) = b'g$, $\beta'(b', g) = b'$, and $(b', g_2)(b'g_2, g_1) = (b', g_2 g_1)$. The natural map $(b', g) \mapsto g$ is again an action morphism. Given any action morphism $\varphi: G' \to G$, $f: B' \to B$ the formula $b'g = \beta'(\varphi^{*-1}(b', g^{-1}))$ defines a right action with $G' \cong f \rtimes G$; indeed this is the right action corresponding to the left action defined by $\varphi: G' \to G$, $f: B' \to B$.

This correspondence between actions and action morphisms is a feature of groupoid theory not available for groups. Whereas a group action can be described diagrammatically only by using the action map $G \times B' \to B'$, which is not a morphism, or the associated $G \to Diff(B')$, which can raise problems of differentiability, a Lie groupoid action can be described in terms of a single morphism of Lie groupoids. This enables us to give conceptually simple proofs in terms of arrows and diagram chasing, especially when passing from a groupoid action to the corresponding Lie algebroid action.

The four classes of morphism listed above may be regarded as giving four notions of surjectivity (or epimorphism) for groupoid morphisms. The fibrations are the largest class for which the kernel $ker(\varphi) = \{g' \in G' \mid \exists b \in B: \varphi(g') = 1_b\}$ is a Lie subgroupoid of $G'$. However, it is not generally possible, given only $G' \rightrightarrows B'$ and the normal subgroupoid $ker(\varphi)$, to construct $G \rightrightarrows B$ and $\varphi$, as is the case with surjective morphisms of groups. The regular fibrations are the morphisms for which this construction is possible, and the result is a straightforward extension of the first isomorphism theorem of elementary algebra ([P3]); for general $s$-fibrations, more elaborate concepts of *kernel system* and *normal subgroupoid system* allow the reconstruction of an $s$-fibration from data given entirely on the domain groupoid ([HM2]). Regular fibrations are also known as *quotient morphisms* or, in [P3], as *extenseurs réguliers*; note that in [P3], the notation $f^*G$ refers to what we have denoted $f^{**}G$.

Inductors can be characterized as regular fibrations whose kernel may be identified with an equivalence relation on the base manifold. This equivalence relation is the kernel pair $R(f) = \{(b', a') \in B' \times B' \mid f(b') = f(a')\}$ of $f$, and the inductor is then equivalent to the morphism in the second pullback diagram above ([P3]).

Turning now to double groupoids, we use the following conventions and notations. A *double groupoid* consists of a quadruple of sets $(S; H, V; B)$, together with groupoid structures on $H$ and $V$, both with base $B$, and two groupoid structures on $S$, a *horizontal structure* with base $V$, amd a *vertical structure* with base $H$, such that the structure maps (source, target, division and identity maps) of each groupoid structure on $S$ are morphisms with respect to the other.

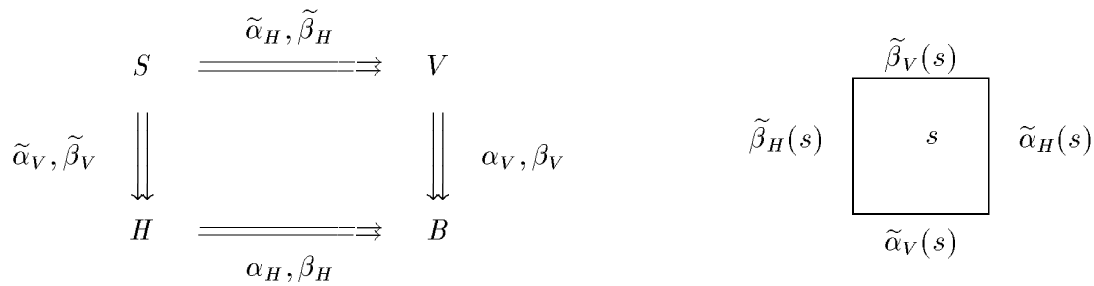
Within $H$ and $V$ we use the multiplicative notation of [M1]. It will normally be clear from the notation for elements which groupoid is under consideration. The identity elements however we denote by $1_b^H \in H$ and $1_b^V \in V$ for $b \in B$. The source, target, anchor, multiplication and division maps of $H$ are denoted $\alpha_H: H \to B$, $\beta_H: H \to B, \chi_H: H \to B \times B$, $\kappa_H: H * H \to H$ and $\delta_H: H \underset{\alpha}{\times} H \to H$, and similarly for $V$, but we will omit the subscripts $H$ and $V$ whenever the meaning is clear.

The two groupoid structures on $S$ we will also write multiplicatively. The horizontal structure with base $V$, denoted $S_H$, will have source and target maps $\widetilde{\alpha}_H: S \to V$, $\widetilde{\beta}_H: S \to V$, anchor $\widetilde{\chi}_H: S \to V \times V$, composition $\widetilde{\kappa}_H: S \underset{V}{*} S \to S$, division $\widetilde{\delta}_H: S \underset{V}{\times} S \to S$, and identities $\widetilde{1}_v^H$ for $v \in V$. The multiplication $\widetilde{\kappa}_H(s_2, s_1)$ we denote by $s_2 \boxminus\!\!\!\!\!\;|\;\, s_1$, and the horizontal inverse of $s$ we denote by $s^{-1(H)}$. For the vertical structure with base $H$, denoted $S_V$, we correspondingly write $\widetilde{\alpha}_V: S \to H$, $\widetilde{\beta}_V: S \to H$ for the source and target projections, $\widetilde{\chi}_V: S \to H \times H$ for the anchor, $\widetilde{\kappa}_V: S \underset{H}{*} S \to S$ for the composition, $\widetilde{\delta}_V: S \underset{H}{\times} S \to S$ for the division, and $\widetilde{1}_h^V$ for $h \in H$ for the identities. The multiplication $\widetilde{\kappa}_V(s_2, s_1)$ we denote by $s_2 \boxminus s_1$, and the vertical inverse of $s$ we denote by $s^{-1(V)}$. For $b \in B$, the *double identity* $\widetilde{1}_{1_b^H}^V = \widetilde{1}_{1_b^V}^H$ is denoted $1_b^2$.

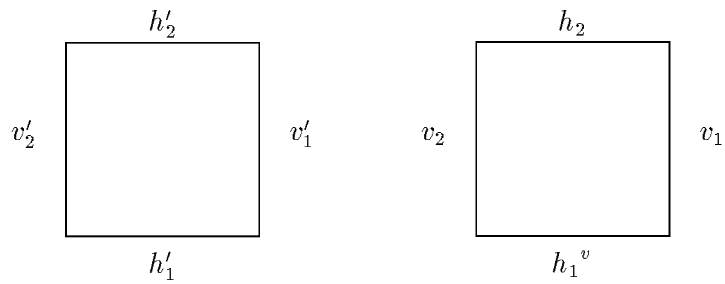
We have used multiplicative notation for all four groupoid structures here in order to reserve additive notation for the associated Lie algebroids; we will frequently need to consider expressions involving a groupoid multiplication in one structure and a Lie algebroid addition in the other.

**Definition 2.1.** *A* double Lie groupoid *is a double groupoid $(S; H, V; B)$ together with differentiable structures on $S$, $H$, $V$ and $B$, such that all four groupoid structures are Lie groupoids and such that the* double source map $\alpha_2: s \mapsto (\widetilde{\alpha}_V(s), \widetilde{\alpha}_H(s))$, $S \to H \underset{\alpha}{\times} V = \{(h, v) | \alpha_H(h) = \alpha_V(v)\}$ *is a surjective submersion.* A morphism of double Lie groupoids $(\varphi; \varphi_H, \varphi_V; \varphi_B): (S'; H', V'; B') \to (S; H, V; B)$ *is a quadruple of smooth maps, $\varphi: S' \to S$, $\varphi_H: H' \to H$, $\varphi_V: V' \to V, \varphi_B: B' \to B$ such that $(\varphi, \varphi_H)$, $(\varphi, \varphi_V)$, $(\varphi_H, \varphi_B)$* and $(\varphi_V, \varphi_B)$ *are morphisms of their respective groupoids.* ■

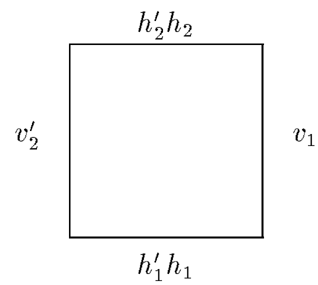
We will often indicate the spaces in a double groupoid and a typical element of it by the diagrams



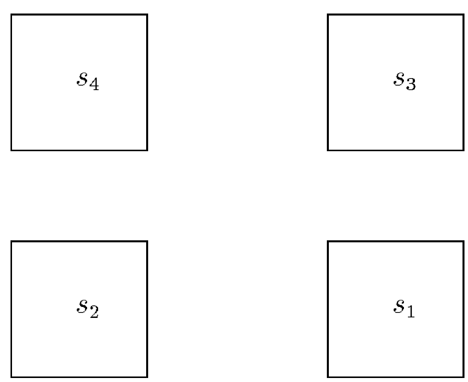
Throughout the paper, diagrams of the latter type are oriented so that the horizontal sides point leftwards and the vertical sides upwards. So a horizontal multiplication of squares




is defined if and only if $v_1' = v_2$, and its sides must be as follows:




Here the two vertical sides are determined by ordinary groupoid axioms in $S_H \Longrightarrow V$, the top side is determined by the condition that $\widetilde{\beta}_V$ is a morphism, and the bottom side by the condition that $\widetilde{\alpha}_V$ is a morphism. The most important axiom is that each division (or multiplication) is a morphism with respect to the other structure; this yields the fact that in




with all inner sides matching, we have $(s_4 \boxminus\!\!\!\!\!\square\; s_3) \boxminus (s_2 \,\square\!\!\!\!\mid\; s_1) = (s_4 \boxminus s_2) \,\square\!\!\!\!\mid\; (s_3 \boxminus s_2)$. This is also known as the interchange law.

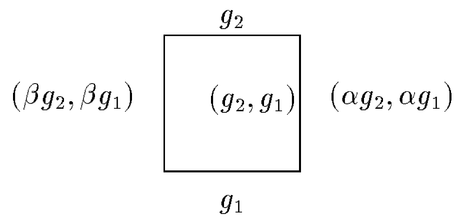
The importance of the condition on the double source map in 2.1 is shown by the following proposition, whose proof is given in [BrM]. It guarantees that the domains of each of the divisions (and multiplications) in $S$ is a Lie subgroupoid of the Cartesian square $S \times S$ of the other structure, so that one may legitimately speak of $S$ as a "Lie groupoid in the category of Lie groupoids". This condition also ensures the easy construction of the double Lie algebroid associated to $S$. However double groupoids with a differentiable structure which satisfy all the conditions of 2.1 except this double source condition have arisen in Lu and Weinstein [LW2].

**Proposition 2.2.** *([BrM, 1.2]) Let $\varphi_1: G_1 \to G$, $f_1: B_1 \to B$ and $\varphi_2: G_2 \to G$, $f_2: B_2 \longrightarrow B$ be fibrations of Lie groupoids such that the pullback manifold $\overline{B}$ of $f_1$ and $f_2$ exists. Then the pullback manifold $\overline{G}$ of $\varphi_1$ and $\varphi_2$ exists and is an embedded Lie subgroupoid of the product groupoid $G_1 \times G_2$. Further, it is the pullback of $\varphi_1$ and $\varphi_2$ in the category of Lie groupoids.*

■

**Example 2.3** (Compare Weinstein [W1, 4.5]; [BrM, 1.3]). Let $G \rightrightarrows B$ be a Lie groupoid. Then the set $G \times G$ can be considered both as the Cartesian product groupoid on base $B \times B$, and as the pair (or coarse) groupoid on base $G$. These two structures constitute a double Lie groupoid

$$\begin{array}{ccc} G \times G & \rightrightarrows & B \times B \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & B \end{array}$$

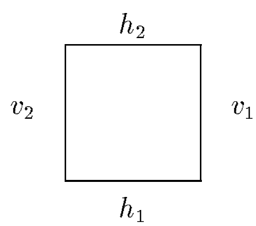



Given any double Lie groupoid $(S; H, V; B)$, the anchor $\widetilde{\chi}_V: S \to H \times H$ together with $id: H \to H$, $\chi_V: V \to B \times B$, $id: B \longrightarrow B$ is a morphism of double groupoids $(S; H, V; B) \to (H \times H; H, B \times B; B)$. Similarly $(\widetilde{\chi}_H; \chi_H, id_V; id_B)$ is a morphism $(S; H, V; B) \to (V \times V; B \times B, V; B)$. ■

**Example 2.4** (Compare [BrM, 1.4]). Let $H$ and $V$ be Lie groupoids on the same base $B$, and suppose that the two anchors $\chi_H: H \to B \times B$ and $\chi_V: V \to B \times B$ are transversal; for example, one may suppose that one or both of $H$ and $V$ are locally trivial. Then the pullback of

$$\begin{array}{ccc} & & V \times V \\ & & \big\downarrow \ \chi_V \times \chi_V \\ H \times H & \xrightarrow[\chi_H \times \chi_H]{} & B^4 \end{array}$$

may be regarded as defining either the pullback groupoid $\chi_H^{**}(V \times V)$ on base $H$ or the pullback groupoid $\chi_V^{**}(H \times H)$ on $V$. These two structures constitute a double Lie groupoid which we denote $\Box\,(H, V)$, and whose elements are squares




with $h_1, h_2 \in H$, $v_1, v_2 \in V$ and sources and targets matching as shown. If $H = V$ we write $\Box H$ for $\Box\,(H, H)$. Taking $H = B \times B$, the pair groupoid on $B$, we obtain the double groupoid $(B^4; B^2, B^2; B)$ in which all four groupoid structures are pair groupoids.

In most of the Ehresmann literature on double groupoids, the notation $\square\, G$ refers to the double groupoid of *commuting* squares in $G$. ■

**Example 2.5** (Compare [BrM, 1.8]). Let $H$ and $V$ be Lie groupoids on a common base $B$, and let $\varphi\colon H \longrightarrow V$ be a morphism over $B$. Then the pullback of

$$\begin{array}{ccc} & & V \\ & & \Big\downarrow \ \chi_V \\ H \times H & \xrightarrow[\alpha_H \times \alpha_H]{\qquad} & B \times B \end{array}$$

may be regarded both as the pullback groupoid $\alpha_H^{**}(V)$ on $H$ and as the action groupoid $(H \times H) \ltimes \chi_V$, where the Cartesian product groupoid $H \times H$ acts on $\chi_V$ by

$$(h_2, h_1)(v) = \varphi(h_2) v \varphi(h_1)^{-1}.$$

These two structures constitute a double Lie groupoid structure, which we denote $\Theta(H, \varphi, V)$ and call the *comma double groupoid* of $\varphi\colon H \to V$. In the case $H = V, \varphi = id$, we write $\Theta(H)$; this structure is often called the *double groupoid of commuting squares* in $H$. Elements of $\Theta(H, \varphi, V)$ have the form

$$\begin{array}{ccc} & h_2 & \\ \varphi(h_2)v\varphi(h_1)^{-1} & \boxed{(h_2, v, h_1)} & v \\ & h_1 & \end{array}$$

■

We briefly recall the main results of [BrM]. Let $(S; H, V; B)$ be a double Lie groupoid, and let $K$ denote the preimage under the double source map $\alpha_2\colon S \to H \underset{\alpha}{\times} V$ of $\{(1_b^H, 1_b^V) \mid b \in B\}$. Then $K$ has a natural groupoid structure on base $B$: the source and target maps are $\alpha_K(k) = \alpha_V(\widetilde{\alpha}_H(k))$, $\beta_K(k) = \beta_H(\widetilde{\beta}_V(k))$, and composition, denoted $\boxbslash$, is defined by

$$k_2 \boxbslash k_1 = (k_2 \boxminus \widetilde{1}_{v_1}^H) \boxbar k_1 = (k_2 \boxbar \widetilde{1}_{h_1}^V) \boxminus k_1$$

where $v_1 = \widetilde{\beta}_H(k_1)$, $h_1 = \widetilde{\beta}_V(k_1)$. The identity of $K$ at $b \in B$ is $1_b^K = 1_b^2$ and the inverse of $k \in K$ is

$$k^{-1(K)} = k^{-1(H)} \boxminus \widetilde{1}_{v^{-1}}^H = k^{-1(V)} \boxbar \widetilde{1}_{h^{-1}}^V ,$$

where $v = \widetilde{\beta}_H(k)$ and $h = \widetilde{\beta}_V(k)$. With respect to this structure, the restrictions of the two target maps

$$\partial_H = \widetilde{\beta}_V : K \to H\,; \qquad \partial_V = \widetilde{\beta}_H : K \to V\,,$$

are morphisms of groupoids over $B$. Further, the kernels of $\partial_H$ and $\partial_V$ commute in the sense that if $m, n \in K$ have $\partial_H(m) = 1_b^H$, $\partial_V(n) = 1_b^V$ for some $b \in B$, then $m \boxbslash n = m \boxbar n = (1_b^2 \boxminus m) \boxbar (n \boxminus 1_b^2) = (1_b^2 \boxbar n) \boxminus (m \boxbar 1_b^2) = n \boxminus m = n \boxbslash m$. We refer to the groupoid $K$ as the *core groupoid* of $(S; H, V; B)$, and $K$ together with the morphisms $\partial_H$ and $\partial_V$, as the *core diagram* of $(S; H, V; B)$.

**Definition 2.6.** *A double Lie groupoid* $(S; H, V; B)$ *is* horizontally locally trivial *if*

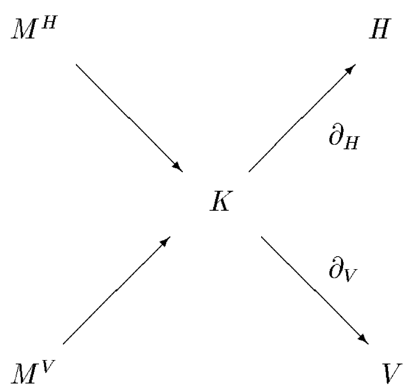
$$\begin{array}{ccc} S_V & \xrightarrow{\ \widetilde{\chi}_H\ } & V \times V \\ \Big\Downarrow & & \Big\Downarrow \\ H & \xrightarrow{\ \chi_H\ } & B \times B \end{array}$$

*is an s-fibration; it is* vertically locally trivial *if* $\widetilde{\chi}_V\colon S_H \to H \times H$, $\chi_V\colon V \to B \times B$ *is an s-fibration; it is* a locally trivial double Lie groupoid *if it is both horizontally and vertically locally trivial.* ∎

These local triviality conditions take account of the double structure. Thus if $(S; H, V; B)$ is horizontally locally trivial, both $H \rightrightarrows B$ and $S_H \rightrightarrows V$ are locally trivial Lie groupoids, but $\widetilde{\chi}_H$ also satisfies the further "smooth filling" condition that it be a fibration with respect to the *vertical* structure on $S$.

**Definition 2.7.** *Let* $H$ *and* $V$ *be locally trivial Lie groupoids on* $B$. *Then* a locally trivial core diagram *for* $H$ *and* $V$ *is a Lie groupoid* $K$ *on* $B$ *(necessarily locally trivial) together with surjective submersions* $\partial_V : K \longrightarrow\!\!\!\!\to V$, $\partial_H : K \longrightarrow\!\!\!\!\to H$ *whose kernels* $M^H = ker(\partial_V)$, $M^V = ker(\partial_H)$ *commute elementwise in* $K$. *If* $H'$ *and* $V'$ *are locally trivial Lie groupoids on* $B'$, *and* $(K', \partial'_H, \partial'_V)$ *is a locally trivial core diagram for* $H'$ *and* $V'$, *then* a morphism of locally trivial core diagrams *is a triple of Lie groupoid morphisms* $\varphi_K : K' \to K$, $\varphi_H : H' \to H$, $\varphi_V : V' \to V$, *all over a map* $\varphi_B : B' \to B$, *such that* $\partial_V \circ \varphi_K = \varphi_V \circ \partial'_V$, *and* $\partial_H \circ \varphi_K = \varphi_H \circ \partial'_H$. *If* $B' = B, H' = H$ *and* $V' = V$, *and if* $\varphi_H, \varphi_V$ *and* $\varphi_B$ *are all identities, then* $\varphi_K$ *is* a morphism of locally trivial core diagrams over $H$ *and* $V$. ∎

A locally trivial core diagram may therefore be represented in the form




(6)

Figure 3.

where $M^H = ker(\partial_V)$ and $M^V = ker(\partial_H)$. Note that $M^H$ and $M^V$ are Lie group bundles, since $\partial_V$ and $\partial_H$ are surjective submersions, and morphisms of Lie groupoids over the fixed base $B$. Further, $M^H$ and $M^V$ commute elementwise in $K$, though neither need itself be commutative.

**Theorem 2.8.** *([BrM, §2]) (i) The core diagram of a locally trivial double Lie groupoid is a locally trivial core diagram. (ii) Given locally trivial Lie groupoids $H$ and $V$ on a common base $B$, and a locally trivial core diagram $(K, \partial_H, \partial_V)$ for $H$ and $V$, there is a locally trivial double Lie groupoid, unique up to isomorphisms which preserve $H$ and $V$, whose core diagram is $(K, \partial_H, \partial_V)$. (iii) These two constructions are mutually inverse equivalences between the categories of locally trivial core diagrams and locally trivial double Lie groupoids.* ■

The proof of (ii) proceeds by quotienting the comma double groupoid $\Theta(K, \partial_V, V)$ over a normal subgroupoid determined by $M^V$.

There are analogues for locally trivial double Lie groupoids of the core sequences (3) and (4). Namely, the local triviality conditions imply that $\widetilde{\alpha}_H\colon S_V \to V$ and $\widetilde{\alpha}_V\colon S_H \to H$ are regular fibrations; that is, that the induced maps $\widetilde{\alpha}_H^{**}\colon S_V \to \alpha_H^{**}V$ and $\widetilde{\alpha}_V^{**}\colon S_H \to \alpha_V^{**}H$ into the pullback groupoids are surjective submersions. These are thus base-preserving morphisms of locally trivial Lie groupoids and yield short exact sequences

$$\beta_H^* M^V \rightarrowtail S_V \xrightarrow{\widetilde{\alpha}_H^{**}} \!\!\!\!\to \alpha_H^{**} V \tag{7}$$

of groupoids over $H$, and

$$\beta_V^* M^H \rightarrowtail S_H \xrightarrow{\widetilde{\alpha}_V^{**}} \!\!\!\!\to \alpha_V^{**} H \tag{8}$$

of groupoids over $V$. Here the kernels are pullbacks of the Lie group bundles $M^V \to B$, $M^H \to B$ and the injections are $(h, m) \mapsto m \boxbar \widetilde{1}_h^V$ and $(v, m) \mapsto m \boxminus \widetilde{1}_v^H$, respectively. In practice, the two exact sequences in (6) are easier to work with and may be regarded as condensed forms of (7) and (8).

The core diagrams of various double Lie groupoids are given in [BrM]. There exist, however, interesting double Lie groupoids whose core groupoid is null. In the remainder of this section we give a complete characterization, in terms quite different from those above, of a large class of such double groupoids. This description arises as a generalization of the work of Lu and Weinstein [LW1] on double Lie groups; see also Majid [Mj].

**Definition 2.9.** *([LW1]) A* double Lie group *is a Lie group $G$ together with two subgroups $H$ and $V$, such that the multiplication map $H \times V \to G$ is a diffeomorphism.* ■

The following theorem is a straightforward generalization of 3.8 of [LW1], and may be proved in the same way. We will obtain a different proof as a consequence of 2.11—2.15.

**Theorem 2.10.** *Let $G$ be a Lie groupoid on $B$ with wide subgroupoids $H, V$ on the same base such that the multiplication $H * V \to G$, $(v, h) \mapsto vh$, where $V * H = \{(v, h) \mid \alpha_V(v) = \beta_H(h)\}$, is a diffeomorphism. For $h \in H, v \in V$ with $\alpha_H(h) = \beta_V(v)$, define $h^v \in H$ and ${}^h v \in V$ by*

$$hv = ({}^h v)(h^v).$$

*Then $(h, v) \mapsto {}^h v$ is a left action of $H$ on $\beta_V\colon V \to B$, and $(h, v) \mapsto h^v$ is a right action of $V$ on $\alpha_H\colon H \to B$ and*

*(i) $\alpha_V({}^h v) = \beta_H(h^v)$ for all $h \in H$, $v \in V$ with $\alpha_H(h) = \beta_V(v)$;*

*(ii)* ${}^{h}(1^V_b) = 1^V_c$ *for* $h \in H$ *with* $\alpha_H(h) = b,\ \beta_H(h) = c$;

*(iii)* $(1^H_c)^v = 1^H_b$ *for* $v \in V$ *with* $\beta_V(v) = c,\ \alpha_V(v) = b$;

*(iv)* ${}^{h}(v_2 v_1) = ({}^{h}v_2)({}^{h^{v_2}}v_1)$ *for* $h \in H$ *and* $v_2,\ v_1 \in V$ *with* $\alpha_H(h) = \beta_V(v_2)$ *and* $\alpha_V(v_2) = \beta_V(v_1)$;

*(v)* $(h_2 h_1)^v = (h_2{}^{{}^{h_1}v})(h_1{}^v)$ *for* $v \in V$ *and* $h_2,\ h_1 \in H$ *with* $\alpha_H(h_2) = \beta_H(h_1)$ *and* $\alpha_H(h_1) = \beta_V(v)$.

*Conversely, let* $H$ *and* $V$ *be Lie groupoids on* $B$ *and suppose that* $(h, v) \mapsto {}^{h}v$ *is a left action of* $H$ *on* $\beta_V$ *and* $(h, v) \mapsto h^v$ *is a right action of* $V$ *on* $\alpha_H$ *such that (i)–(v) hold. Then* $V * H$ *has a groupoid structure on* $B$ *given by*

$$\beta(v,h) = \beta_V(v), \qquad \alpha(v,h) = \alpha_H(h)$$

$$(v_2,h_2)(v_1,h_1) = (v_2({}^{h_2}v_1), (h_2^{v_1})h_1)$$

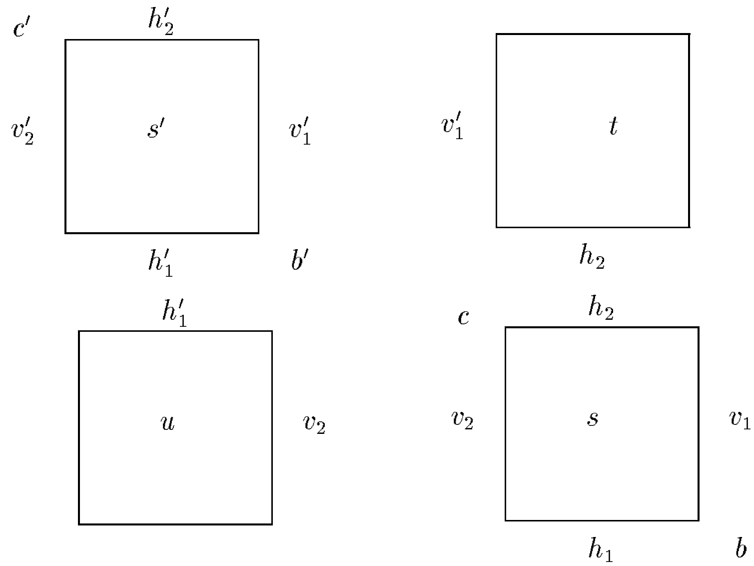
*and the canonical maps* $H \to V * H$, $V \to H * V$ *represent* $H$ *and* $V$ *as closed embedded wide subgroupoids of* $V * H$.

*These constructions are mutually inverse.* ∎

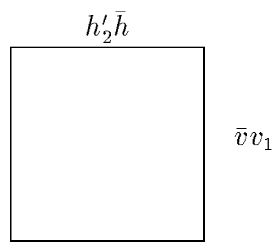
A subgroupoid is *wide* if it has the same base as the ambient groupoid. Double Lie groups have been introduced by a number of authors; see the references in [LW1] and [Mj]. We now show that double Lie groups, and the groupoid version of them in 2.10, admit a double Lie groupoid structure in the sense of 2.1. In fact we show that double Lie groupoids which arise in this way may be characterized as follows.

**Definition 2.11.** *A double Lie groupoid* $(S; H, V; B)$ *is* vacant *if the double source map* $\alpha_2\colon S \to H \underset{\alpha}{\times} V$ *is a diffeomorphism.* ∎

In a vacant double groupoid $S$, any element $s$ is determined by any two consecutive sides. Thanks to this, we will define a third groupoid structure on $S$, with base $B$, which we will denote $S_D$ and call the *diagonal groupoid structure* of $S$, the terminology having been introduced in a special case by Kock ([K], §2). The source and target maps of $S_D$ are $\alpha_D = \alpha_H \circ \widetilde{\alpha}_V$ and $\beta_D = \beta_H \circ \widetilde{\beta}_V$, and the multiplication $s' \diamond s$ is defined in the diagram below: given $s'$ and $s$ as shown, with $b' = c$,

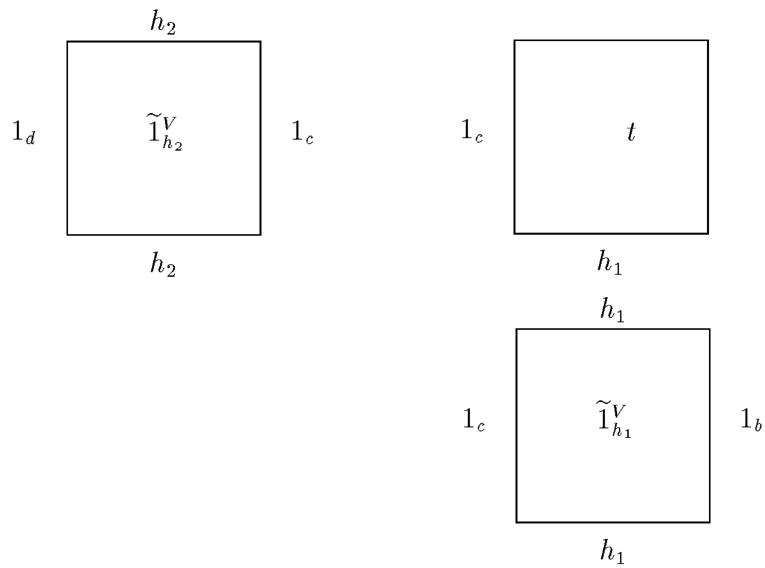



there exists a unique square $t \in S$ with sides $v_1'$ and $h_2$ as shown. Let $\bar{h} = \widetilde{\beta}_V(t)$ and $\bar{v} = \widetilde{\alpha}_H(t)$ be the other two sides of $t$, and define $s' \diamond s$ to be the unique square with sides

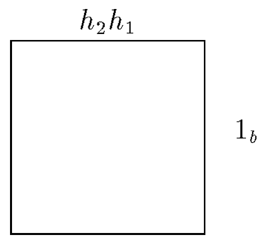



By reasoning diagrammatically, one can see that use of the square $u$ instead of $t$ yields the same value for $s' \diamond s$, and that this composition makes $S_D$ a groupoid on $B$; that it is a Lie groupoid follows easily. The identity $1_b^D$ for $b \in B$ is $1_b^2$, and the inverse $s^{-1(D)}$ is $(s^{-1(H)})^{-1(V)} = (s^{-1(V)})^{-1(H)}$.

The map $\widetilde{1}^V\colon H \to S$ now becomes a morphism of groupoids over $B$, from $H$ into $S_D$, as the diagram

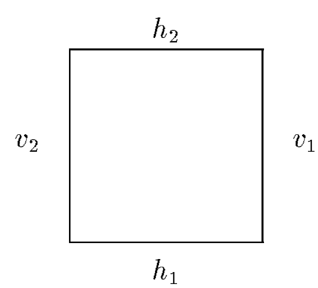



where $d = \beta(h_2)$, $c = \alpha(h_2) = \beta(h_1)$ and $b = \alpha(h_1)$, shows; the only possible value for $t$ is $\widetilde{1}^V_{h_1}$ itself, and so $\widetilde{1}^V_{h_2} \diamond \widetilde{1}^V_{h_1}$ has the sides



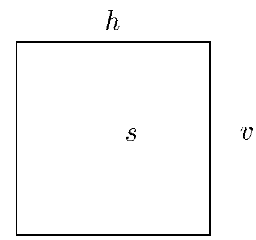

and therefore must be $\widetilde{1}^V_{h_2h_1}$.

Since $\widetilde{1}^V : H \to S$ is a closed embedding, we may consider $H$ to be a closed embedded Lie subgroupoid of $S_D$. In the same way, $\widetilde{1}^H : V \to S$ represents $V$ as a closed embedded Lie subgroupoid of $S_D$. Further, any element



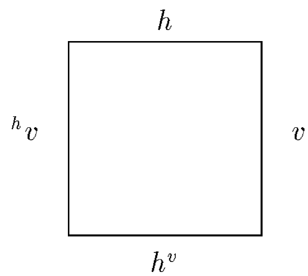

can be written as $h_2 \diamond v_1$, or as $v_2 \diamond h_1$, and these are unique representations of $s$ as products of elements from $H$ and $V$ in the indicated orders.

Associated with the vacant double Lie groupoid $(S; H, V; B)$ are two actions, of $H$ on $V$ and of $V$ on $H$. Namely, given $h \in H$, $v \in V$ with $\alpha(h) = \beta(v)$, form the unique square $s$ with sides $h$ and $v$ as indicated




and define ${}^h v = \widetilde{\beta}_H(s)$ and $h^v = \widetilde{\alpha}_V(s)$. We will see that this defines a left action of $H$ on $\beta_V : V \to B$, and a right action of $V$ on $\alpha_H : H \to B$. (These actions may also be obtained by observing that the vacancy condition 2.11 forces both $(\widetilde{\beta}_V, \beta_V)$ and $(\widetilde{\alpha}_H, \alpha_H)$ to be action morphisms.)

**Proposition 2.12.** *Given a vacant double Lie groupoid $(S; H, V; B)$, the diagram*

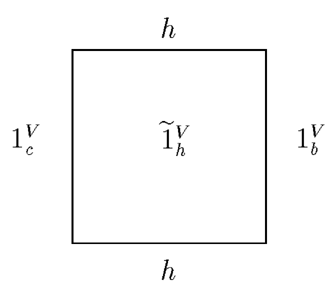


(9)

*defines a left action of $H$ on $\beta_V$ and a right action of $V$ on $\alpha_H$, such that conditions (i)—(v) of 2.10 hold.*

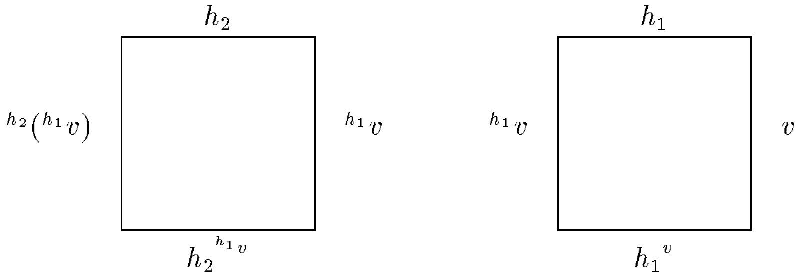
*Further, $S_H$ is naturally isomorphic to $H \ltimes \beta_V$ and $S_V$ is naturally isomorphic to $\alpha_H \rtimes V$.*

PROOF: That $\beta_V({}^h v) = \beta_H(h)$, that $\alpha_H(h^v) = \alpha_V(v)$, and that $\alpha_V({}^h v) = \beta_H(h^v)$, where $\alpha_H(h) = \beta_V(v)$, all follow from diagram (9). Next, consider the square

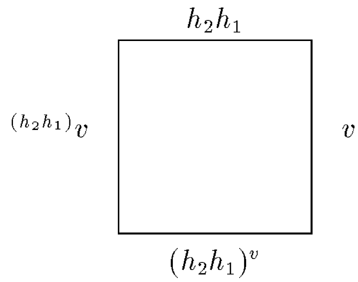



where $h \in H^c_b$. Comparing this with (9) shows both that $h^{1^V_b} = h$ and that (ii) is satisfied. In a similar way, the square $\widetilde{1}^H_v$, for $v \in V^c_b$, shows that ${}^{1^H_c} v = v$ and that (iii) is satisfied.

Lastly, consider the following two squares, where $h_2$, $h_1$, $v$ are as in (v). Form the right-hand square first.




Their composite must be




Equating the two expressions for the composite shows that ${}^{h_2}({}^{h_1}v) = {}^{(h_2h_1)}v$, and that (v) is satisfied. A similar pair of squares, and their vertical composition, establish the remaining equations.

That $S_H \cong H \ltimes \beta_V$, and that $S_V \cong \alpha_H \rtimes V$, follow immediately. ∎

**Theorem 2.13.** *Let $H$ and $V$ be Lie groupoids on base $B$. Suppose given a left action $(h, v) \mapsto {}^h v$ of $H$ on $\beta_V$, and a right action $(h, v) \mapsto h^v$ of $V$ on $\alpha_H$, such that equations (i)—(v) of 2.10 hold. Then there is a vacant double Lie groupoid $(S; H, V; B)$, unique up to isomorphisms preserving $H$ and $V$, such that the actions of $H$ and $V$ induced by $S$ as in 2.12 are exactly the given ones.*

PROOF: The pullback manifold $S = H * V$ of $\alpha_H$ and $\beta_V$ supports both the left action groupoid structure $H \ltimes \beta_V$ and the right action groupoid structure $\alpha_H \rtimes V$. It is straightforward to verify that these structures make $(S; H, V; B)$ a vacant double Lie groupoid with the specified properties. The uniqueness follows from the last statement in 2.12. ∎

**Definition 2.14.** *Let $H$ and $V$ be Lie groupoids on the same base $B$. Then an* interaction *of $H$* with *$V$ consists of a left action of $H$ on $\beta_V$ and a right action of $V$ on $\alpha_H$, satisfying (i)—(v) of 2.10.* ∎

In fact, equations (ii) and (iii) of 2.10 follow from (i), (iv) and (v). I am grateful to J.-H. Lu for this remark. There is now the following summary theorem.

**Theorem 2.15.** *Let $H$ and $V$ be Lie groupoids on a common base $B$. Then 2.12 and 2.13 give a bijective correspondence between interactions of $H$ with $V$ and vacant double Lie groupoids $(S; H, V; B)$.* ∎

The vacant double Lie groupoid corresponding to an interacting pair of Lie groupoids $H$ and $V$ we denote $H \bowtie V$, and call the *indirect product* of $H$ and $V$, following a suggestion of A. Weinstein. The bowtie notation differs slightly from that of [LW1], [Mj], where it refers only to (what we call) the diagonal structure.

There is now an alternative proof of the second part of 2.10. Given Lie groupoids $H$ and $V$ and an interaction between them, take the indirect product $H \bowtie V$ and let $G$ denote the diagonal groupoid structure $(H \bowtie V)_D \rightrightarrows B$. Then $H$ and $V$ are canonically embedded as subgroupoids of $G$ and if the manifold underlying $G$ is regarded as $V * H$, then the diagonal groupoid structure is precisely the groupoid structure defined in 2.10.

# 3 PRINCIPAL DOUBLE GROUPOIDS: AFFINOIDS AND PREGROUPOIDS

Double Lie groups have been characterized in §2 as vacant double groupoids whose side groupoids are groups. At the other extreme are vacant double groupoids whose side groupoids are equivalence relations. These are precisely the *pregroupoids* of Kock [K] and the *affinoid structures* of Weinstein [W2]; see these papers for earlier references. Here we call them principal double groupoids and use the diagonal structure of §2 to establish correspondences between principal double groupoids and what may be called "principal bundles with structure Lie groupoid" (Kock [K]) and between principal double groupoids and the "butterfly" diagrams of Pradines [P2]. Both these latter concepts involve only ordinary groupoids: a principal bundle with structure Lie groupoid arises from a free action of a Lie groupoid in the same way as an ordinary principal bundle arises from a free action of a Lie group; and butterfly diagrams are diagrams of ordinary groupoids which may be regarded as giving a strong form of Morita equivalence for Lie groupoids. These correspondences will thus enable us, in the second part of the paper, to test the concept of double Lie algebroid in cases where a natural answer is provided by ordinary groupoid theory.

We first need concepts of transitivity for double groupoids. If $G \rightrightarrows B$ is a set-theoretic groupoid then $b_1 \sim b_2 \Leftrightarrow \exists g \in G{:}\, \beta(g) = b_2, \alpha(g) = b_1$ defines an equivalence relation on $B$, the *transitivity relation*, whose classes $< g >$ are the *transitivity components* of $G$; we denote the set of equivalence classes by $\tau_0(G)$ and the projection $g \mapsto < g >$ by $t_G{:}\, B \to \tau_0(G)$. If $G$ is a Lie groupoid on $B$ the transitivity components are submanifolds ([M1, III 1.6]) but $\tau_0(G)$ is not usually a quotient manifold of $B$.

Now consider a set-theoretic double groupoid $(S; H, V; B)$, and assume that the double source map is a surjection. The side groupoids $H$ and $V$ define transitivity relations on $B$ with quotient projections $t_H{:}\, B \to \tau_0(H)$ and $t_V{:}\, B \to \tau_0(V)$. There is also a transitivity relation on $V$ defined by the horizontal structure of $S$, and the corresponding projection $t_{S_H}{:}\, V \to \tau_0(S_H)$ we denote by $\widetilde{t}_H{:}\, V \to \tau_1^H(S),\ v \mapsto < v >$.

**Proposition 3.1.** *With the above notation, there is a unique groupoid structure on $\tau_1^H(S)$ with base $\tau_0(H)$ such that the natural projections $\widetilde{t}_H{:}\, V \to \tau_1^H(S), t_H{:}\, B \to \tau_0(H)$ form a morphism of groupoids; this morphism is a fibration of groupoids.*

PROOF: Define source and target maps by $\alpha(< v >) = < \alpha_V(v) >,\ \beta(< v >) = < \beta_V(v) >$. Given $< v_2 >, < v_1 >$ with $\alpha(< v_2 >) = \beta(< v_1 >)$, there exists $h \in H$ with $\alpha(v_2) = \beta(h),\ \alpha(h) = \beta(v_1)$. Take any $s \in S$ with $\widetilde{\beta}_V(s) = h,\ \widetilde{\alpha}_H(s) = v_1$, and define $< v_2 >< v_1 > = < v_2 \widetilde{\beta}_H(s) >$. The double source condition ensures that such an $s$ exists. It is straightforward to check that this gives a well-defined groupoid structure, and that the natural projections form a fibration. Uniqueness is immediate. ∎

This groupoid $\tau_1^H(S) \rightrightarrows \tau_0(H)$ is the *horizontal transitivity groupoid* of $S$. There is also of course a *vertical transitivity groupoid* $\tau_1^V(S)$ on $\tau_0(V)$. Note that $\tau_1^H(S) \Longrightarrow \tau_0(H)$ is not a quotient groupoid of $V \Longrightarrow B$ in the usual meaning of the term, but is so in the extended sense of [HM2]. The corresponding congruence on $V$ is formed from the images of the two anchors $\widetilde{\chi}_H: S \to V \times V$ and $\chi_H: H \to B \times B$.

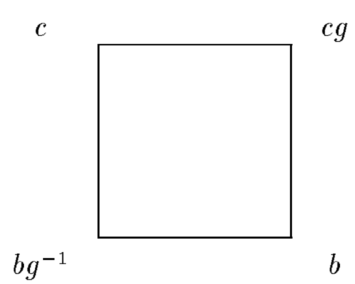
If $S$ is a double Lie groupoid neither $\tau_1^H(S)$ nor $\tau_0(H)$ need inherit quotient manifold structures. However if they do, and if $\chi_H: H \to im(\chi_H)$ is a surjective submersion, then it follows from [HM2] that $\tau_1^H(S)$ is a Lie groupoid on $\tau_0(H)$ and that the natural projections form a fibration of Lie groupoids.

**Definition 3.2.** *Let $B$ be a manifold, and let $p_1: B \to Q_1$ and $p_2: B \to Q_2$ be surjective submersions. A* principal double Lie groupoid *on $(B, p_2, p_1)$ is a vacant double Lie subgroupoid $(S; H, V; B)$ of $(B^4; B^2, B^2; B)$ in which $H$ is the kernel pair groupoid $R(p_2)$ and $V$ is the kernel pair groupoid $R(p_1)$.* ■

Double groupoids of this type were studied, in the context of synthetic differential geometry, by Kock [K], who called them *pregroupoids*, and, in Poisson geometry, by Weinstein [W2], who called them *affinoid structures.* Forms of this concept go back a long way; see the references in [K] and [W2]. We use the term "principal double Lie groupoid" in order to identify these structures as a class of double groupoids; this can perhaps be justified by analogy with the case of ordinary groupoids, where a principal groupoid is one in which any two points of the base are joined by at most one arrow, and by the following example.

**Example 3.3.** [K, §2] Let $G_1$ be a Lie groupoid on base $Q_1$, and let $p_1: B \to Q_1$ be a surjective submersion on which $G_1$ acts to the right by $(b, g) \mapsto bg$. Suppose, further, that this action is *free* in the sense that for $b \in B$ and $g, g' \in G_1$ with $p_1(b) = \beta(g) = \beta(g')$, if $bg = bg'$ then $g = g'$; and suppose also that the orbit manifold $Q_2 = B/G_1$ exists, with quotient map $p_2: B \to Q_2$. We will refer to this situation, denoted $B(Q_2, G_1, p_2)(Q_1, p_1)$, as a *principal bundle with structure Lie groupoid* (compare Pradines [P2]).

Within the double Lie groupoid $(B^4; B^2, B^2; B)$ of 2.4, let $S$ be the set of all squares of the form




where $b, c \in B, g \in G_1$ and $p_1(c) = \beta(g), p_1(b) = \alpha(g)$. Denote this square by $(bg^{-1}, c, cg, b)$. Then $(S; R(p_2), R(p_1); B)$ is a principal double Lie groupoid on $(B, p_2, p_1)$.

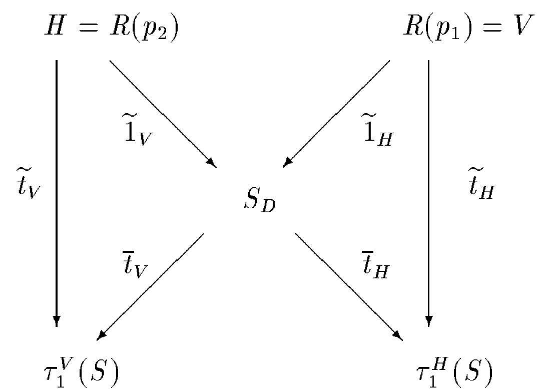
Observe that the diagonal groupoid $S_D$ is isomorphic to the pullback groupoid $p_1^{**}G_1 \Longrightarrow B$ under the correspondence $(bg^{-1}, c, cg, b) \leftrightarrow (c, g, b)$. ■

Let $(S; H, V; B)$ be a vacant double groupoid and neglect, for the moment, any questions of differentiability. Then the fibrations $\widetilde{t}_H: V \to \tau_1^H(S)$ and $\widetilde{t}_V: H \to \tau_1^V(S)$ may be lifted to $S$ and, with respect to the diagonal structure, give morphisms $\bar{t}_H: S_D \to \tau_1^H(S), s \mapsto \, < \widetilde{\alpha}_H(s) >$ $= \, < \widetilde{\beta}_H(s) >$ and $\bar{t}_V: S_D \to \tau_1^V(S), s \mapsto \, < \widetilde{\alpha}_V(s) > \, = \, < \widetilde{\beta}_V(s) >$, which are still fibrations.

If now $(S; R(p_2), R(p_1); B)$ is a principal double Lie groupoid, these two fibrations become inductors, the kernel of $\bar{t}_H$ being $H = R(p_2)$, embedded in $S_D$ via $\widetilde{1}_V$, and the kernel of $\bar{t}_V$ being $V = R(p_1)$. In other words, $t_H: B \to \tau_0(H)$ identifies with $p_2: B \to Q_2$, and $t_V: B \to \tau_0(V)$ with $p_1: B \to Q_1$ and (in the notation from the start of §2) the induced maps $\bar{t}_H^{**}: S_D \to p_2^{**}\tau_1^H(S)$ and $\bar{t}_V^{**}: S_D \to p_1^{**}\tau_1^V(S)$ are diffeomorphisms. Further, the left action of $H$ on $\beta_V: V \to B$ described in 2.11 quotients to an action (which we will take to be a right action) of $\tau_1^V(S)$ on $p_1: B \to Q_1$, namely

$$b < h > = < \alpha_V({}^{h^{-1}}(\beta_H(h), b)) >$$

where $p_1(b) = \beta(< h >) = p_1(\beta_H(h))$. Similarly the right action of $V$ on $\alpha_H: H \to B$ from 2.11 quotients to an action (which we will take to be a left action) of $\tau_1^H(S)$ on $p_2: B \to Q_2$. Altogether we have the diagram in Figure 4.



(10)

Figure 4.

It is easy to verify that (10) is a butterfly diagram in the following sense of Pradines [P2].

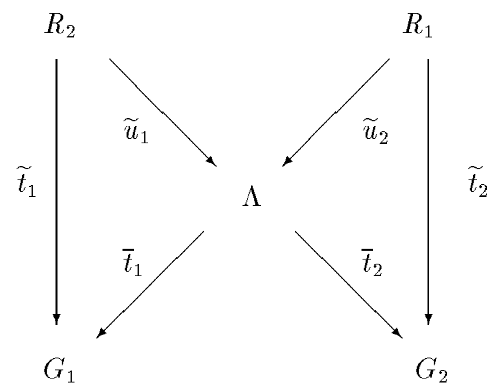
**Definition 3.4.** *Let $B$ be a manifold, and let $p_1: B \to Q_1$ and $p_2: B \to Q_2$ be surjective submersions. Then* a butterfly diagram *on $(B, p_2, p_1)$ is a commutative diagram of Lie groupoids of the form of Figure 5, where $\widetilde{u}_2, \widetilde{u}_1$ are closed embeddings of Lie groupoids over $B$, where $\bar{t}_2, \bar{t}_1$ are inductors over $p_2: B \to Q_2$ and $p_1: B \to Q_1$ respectively, where $\widetilde{t}_2, \widetilde{t}_1$ are action morphisms over $p_2$ and $p_1$, and where the two diagonal sequences are exact at $\Lambda$.* ∎

The effect of the exactness conditions is to force $R_2 = R(p_2)$ and $R_1 = R(p_1)$. If $(S; R(p_2), R(p_1); B)$ is the principal double groupoid arising in 3.3 from $B(Q_2, G_1, p_2)(Q_1, p_1)$ then (10) takes the form of Figure 6.

Here $G_2$ is the orbit manifold of $R(p_1)$ under the diagonal action $(c, b)g = (cg, bg)$, defined if $p_1(c) = p_1(b) = \beta(g)$. Denote the orbit of $(c, b)$ by $< c, b >$. The groupoid structure on $G_2$ is given by $\beta_2(< c, b >) = p_2(c), \alpha_2(< c, b >) = p_2(b)$ and

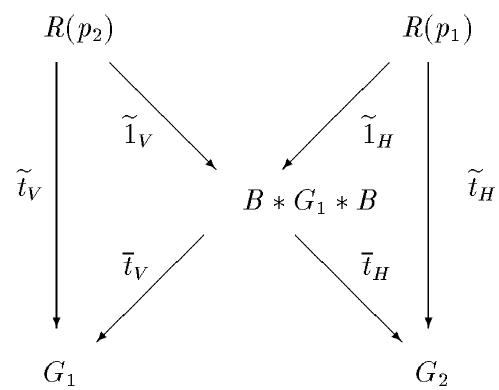
$$< c', b' > < c, b > = < c', bg >$$

where $b' = cg, g \in G_1$. The notation $B * G_1 * B$ indicates the pullback groupoid $p_1^{**}G_1$, written symmetrically as $\{(c, g, b) \in B \times G_1 \times B | p_1(c) = \beta(g), \alpha(g) = p_1(b)\}$.




(11)

Figure 5.




(12)

Figure 6.

The maps are given by $\widetilde{t}_V(c, cg) = g, \widetilde{t}_H(c, b) = < c, b >, \widetilde{1}_V(c, cg) = (c, g, cg), \widetilde{1}_H(c, b) = (c, 1, b)$ and $\bar{t}_V(c, g, b) = g, \bar{t}_H(c, g, b) = < cg, b >$. The (right) action of $G_1$ on $p_1 \colon B \to Q_1$ is the bundle action, and the (left) action of $G_2$ on $p_2 \colon B \to Q_2$ is $< c, b > (b') = cg$, where $b' = bg, g \in G_1$. The conditions of 3.4 may be checked directly in these terms.

The construction of $G_2$ from $B(Q_2, G_1, p_2)(Q_1, p_1)$ thus generalizes the construction from an ordinary principal bundle $B(Q, G, p)$ of the associated *gauge* (or *Ehresmann*) *groupoid* $\frac{B \times B}{G} \rightrightarrows Q$ (see, for example, [M1, I 1.10]). One may prove that, in the general case, the categories of actions of $G_1$ and $G_2$ can be identified, and so $G_1$ and $G_2$ in 3.4 may be said to be Morita equivalent ([K, §3], see also [X] for a form appropiate to symplectic groupoids).

We now complete this circle of ideas by showing that associated with every butterfly diagram is a principal bundle with structure Lie groupoid. Consider diagram (11) in 3.4. Take the action of $G_1$ on $p_1 \colon B \to Q_1$ to be a right action, namely $bg = \alpha_H((\widetilde{t}_V^*)^{-1}(b, g))$. That this is a free action follows from the fact that the action groupoid $H = R(p_2)$ is an equivalence relation on $B$; more precisely, the anchor of $H$ embeds it in $B \times B$ as a subgroupoid. The orbits of the action are (of course) the transitivity components of $H$ and since $H = R(p_2)$, the orbit manifold is $Q_2$ with $p_2$ the projection. Thus $B(Q_2, G_1, p_2)(Q_1, p_1)$ is a principal bundle with structure Lie groupoid.

We now have three constructions: of a principal double Lie groupoid from a principal bundle with structure Lie groupoid; of a butterfly diagram from a principal double Lie groupoid; and of a principal bundle with structure Lie groupoid from a butterfly diagram. The proof of the following result is now straightforward (compare [K], [P2]).

**Theorem 3.5.** *The three constructions described above give a commuting triangle of equivalences between the concepts of principal bundle with structure Lie groupoid, principal double Lie groupoid, and butterfly diagram.* ∎

Of the three reverse constructions, the only one we need describe is that from butterfly diagram to principal double Lie groupoid. With the notation of 3.4, let $(b, g) \mapsto bg$ denote the right action of $G_1$ on $p_1 \colon B \to Q_1$. Take the corresponding left diagonal action on $R(p_1) \to Q_1$, namely $g(c, b) = (cg^{-1}, bg^{-1})$. Now define a left action of $H = R(p_2)$ on $\beta_V \colon V = R(p_1) \to B$ by ${}^h(c, b) = g(c, b)$, where $g = \widetilde{t}_1(h)$. Similarly one can define a right action of $V$ on $\alpha_H \colon H = R(p_2) \to B$, and these form an interaction for $H$ and $V$ in the sense of 2.12, and the corresponding vacant double Lie groupoid is the one required.

# 4 $\mathcal{LA}$-GROUPOIDS

$\mathcal{LA}$-groupoids arise from a single application of the Lie functor to a double Lie groupoid, and represent an intermediate stage between double Lie groupoids and double Lie algebroids. We need to begin with a concept of $\mathcal{VB}$-groupoid, introduced by Pradines [P4]: a *$\mathcal{VB}$-groupoid* is a Lie groupoid in the category of vector bundles. In detail, a $\mathcal{VB}$-groupoid is a diagram

$$\begin{array}{ccc} \Omega & \overset{\widetilde{\alpha},\ \widetilde{\beta}}{\Longrightarrow} & A \\ {\scriptstyle \widetilde{q}}\downarrow & & \downarrow{\scriptstyle q} \\ G & \underset{\alpha,\ \beta}{\Longrightarrow} & B \end{array}$$

in which $G$ is a Lie groupoid on $B$, $A$ is a vector bundle on $B$, and $\Omega$ has both a Lie groupoid structure on $A$, and a vector bundle structure on $G$, in such a way that the structure maps of the groupoid structure on $\Omega$ are vector bundle morphisms—or, equivalently, in such a way that the structure maps of the vector bundle structure are groupoid morphisms—and, finally, such that the map

$$(\widetilde{q}, \widetilde{\alpha}) \colon \Omega \to G \underset{B}{\times} A = \{(g, X) \in G \times A \mid \alpha(g) = q(X)\}$$

is a surjective submersion. We refer to $(\widetilde{q}, \widetilde{\alpha})$, abusively, as the *double source map* of $(\Omega; G, A; B)$. Clearly a $\mathcal{VB}$-groupoid is a double Lie groupoid for which one structure is a vector bundle.

The groupoid maps in $\Omega$ we denote by $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\delta}$, $\widetilde{\kappa}$, $\widetilde{1}$ and $\widetilde{\chi}$, with composition denoted by juxtaposition, identities $\widetilde{1}_X$ for $X \in A$, and inverses $\xi^{-1}$ for $\xi \in \Omega$. In the vector bundle structure on $\Omega$ we denote addition, scalar multiplication and subtraction by the usual symbols, but the zero above $g \in G$ will be $\widetilde{0}_g$. In $A$ and $G$ we use the usual notations. The module of sections of the vector bundle $(\Omega, \widetilde{q}, G)$ is denoted $\Gamma_G \Omega$.

In the same way, an $\mathcal{LA}$-groupoid is a Lie groupoid in the category of Lie algebroids. We first recall some basic definitions. A *Lie algebroid on base* $B$ is a vector bundle $(A, q, B)$ together with a map $a: A \to TB$ of vector bundles over $B$, the *anchor*, and an $\mathbf{R}$-bilinear, antisymmetric bracket of sections $[\ ,\ ]: \Gamma A \times \Gamma A \to \Gamma A$, which obeys the Jacobi identity and is such that (i) $a([X, Y]) = [a(X), a(Y)]$, and (ii) $[X, uY] = u[X, Y] + a(X)(u)Y$ for $X, Y \in \Gamma A, u \in C(B)$. Here $C(B)$ is the ring of smooth functions $B \to \mathbf{R}$ and $a(X)(u)$ is the Lie derivative of $u$ with respect to the vector field $a(X)$. A Lie algebroid $A$ is said to be *transitive* if $a: A \to TB$ is surjective, and *totally intransitive* if $a = 0$. We will often denote a Lie algebroid as $a: A \to TB$.

Suppose that $a': A' \to TB'$ and $a: A \to TB$ are Lie algebroids, and that $\varphi: A' \to A$, $f: B' \to B$ is a morphism of the underlying vector bundles. Recall ([HM1, §1]) that $X' \in \Gamma A'$ and $X \in \Gamma A$ are said to be *$\varphi$-related* if $\varphi \circ X' = X \circ f$; we also then say that $X'$ is *$\varphi$-projectable* (and *projects to* $X$), and write $X' \overset{\varphi}{\sim} X$. The general definition of a Lie algebroid morphism is given in [HM1, §1], but the following special case will suffice in this paper.

**Proposition 4.1.** *([HM1, 1.5]) With the above notation, if $\varphi$ is fibrewise surjective, then it is a morphism of Lie algebroids over $f$ iff $a \circ \varphi = T(f) \circ a'$ and $X' \overset{\varphi}{\sim} X$, $Y' \overset{\varphi}{\sim} Y \implies [X', Y'] \overset{\varphi}{\sim} [X, Y]$.* ∎

Note that in [HM1], the action of $X \in \Gamma A$ on $u \in C(B)$ given by $a(X)(u)$ is denoted $[X, u]$. In [HM1, §1] can also be found the definition of a product of Lie algebroids, of Lie subalgebroid, and other basic material. Given a Lie groupoid $G \rightrightarrows B$, the tangent bundle along the $\alpha$-fibres, $T^\alpha G = kerT(\alpha)$, pulled back along the identity map, $1: B \to G$, is the *Lie algebroid $AG$ of $G$*. The bracket structure is obtained by identifying sections of $AG$ with right-invariant (and $\alpha$-vertical) vector fields on $G$, as with the Lie algebra of a Lie group. A right-invariant and $\alpha$-vertical vector field $X$ on $G$ is projectable under $\beta$ to a vector field $a(X)$ on $B$ and this defines the anchor of $AG$. A morphism of Lie groupoids $\varphi: G' \to G$, $f: B' \to B$ induces a morphism of the Lie algebroids $A(\varphi): AG' \to AG$ over $f$ in the obvious way. These constructions constitute the *Lie functor* from the category of Lie groupoids to that of Lie algebroids. For further details see [HM1, §1] and [M1, Chapter III, §3]. An alternative construction of the Lie algebroid of a Lie groupoid in terms of the normal bundle to $B$ within $G$ is given in [CDW].

The four classes of groupoid morphism described at the start of §2 have analogues for Lie algebroids. Consider a morphism of Lie algebroids $\varphi: A' \longrightarrow A$, $f: B' \longrightarrow B$, and form, as before, the pullback manifold

$$\begin{array}{ccc} f^*A & \longrightarrow & A \\ \Big\downarrow & & \Big\downarrow q \\ B' & \underset{f}{\longrightarrow} & B. \end{array}$$

Let $\varphi^*\colon A' \to f^*A$ be the induced map $X' \mapsto (q'(X'), \varphi(X'))$. Then $(\varphi, f)$ is a *fibration* if $\varphi^*$ is a surjective submersion, and $(\varphi, f)$ is an *action morphism* if $\varphi^*$ is a diffeomorphism. If $f\colon B' \to B$ is also a surjective submersion then we speak of an *s-fibration* or an *s-action morphism.*

When $f$ is a surjective submersion one can also form the pullback (which is in fact the pullback Lie algebroid)

$$\begin{array}{ccc} f^{**}A & \longrightarrow & A \\ \Big\downarrow & & \Big\downarrow a \\ TB' & \underset{T(f)}{\longrightarrow} & TB. \end{array}$$

There is now an induced map $\varphi^{**}\colon A' \to f^{**}A,\ X' \mapsto (a'(X'), \varphi(X'))$. In this case, $(\varphi, f)$ is a *regular fibration* if $\varphi^{**}$ is a surjective submersion, and is an *inductor* if $\varphi^{**}$ is a diffeomorphism.

Clearly if a morphism of Lie groupoids $\varphi\colon G' \to G,\ f\colon B' \to B$ is an action morphism, fibration, regular fibration or inductor, then the induced morphism $A(\varphi)\colon AG' \to AG$ has the same property. Further, these four classes of Lie algebroid morphism correspond to the same basic algebraic constructions of action, quotient and pullback as is the case for groupoids ([HM1, §2—§4], [HM2, 3.4]). We briefly recall the details in the case of actions.

Let $A$ be a Lie algebroid on $B$ and let $f\colon B' \to B$ be a smooth map. Then an *action* of $A$ on $f\colon B' \to B$ is an $\mathbf{R}$-linear map $X \mapsto X^*$, $\Gamma A \to \Gamma TB'$ such that (i) $(uX)^* = (u \circ f)X^*$ for $X \in \Gamma A,\ u\colon B \to \mathbf{R}$; (ii) $[X,Y]^* = [X^*, Y^*]$ for $X, Y \in \Gamma A$; (iii) $X^* \overset{f}{\sim} a(X)$ for $X \in \Gamma A$. (Thanks to (i), the action may be regarded as a morphism $f^*A \to TB'$, $(b', X) \mapsto X^*(b')$ of vector bundles over $B'$.) It is shown in [HM1, §2] that actions of $A$ on $f\colon B' \to B$ are in bijective correspondence with action morphisms of Lie algebroids over $f$ and with codomain $A$. Namely, given an action of $A$ on $f$, define a Lie algebroid structure on the pullback vector bundle $f^*A$ with anchor $a'(b', X) = X^*(b')$ and bracket $\left[\sum u'_i \otimes X_i, \sum v'_j \otimes Y_j\right] = \sum u'_i v'_j \otimes [X_i, Y_j] + \sum u'_i X^*_i(v'_j) \otimes Y_j - \sum v'_j Y^*_j(u'_i) \otimes X_i$, where $\Gamma(f^*A)$ is regarded as $C(B') \underset{C(B)}{\otimes} \Gamma A$. With this structure $f^*A$ is denoted $A \ltimes f$ or $A \ltimes B'$ and called the *action* or *transformation Lie algebroid*; the canonical map $A \ltimes f \to A$ is an action morphism of Lie algebroids. Conversely, if $\varphi\colon A' \to A,\ f\colon B' \to B$ is an action morphism of Lie algebroids, then $A' \cong A \ltimes f$, where $A$ acts on $f$ by $X^*(b') = a'((\varphi^*)^{-1}(b', X))$.

If a Lie groupoid $G \rightrightarrows B$ acts on $f\colon B' \to B$ then $A(G \ltimes f) \cong AG \ltimes f$, where the induced action of $AG$ on $f$ is $X^*(b') = T(g \mapsto gb')_1(X(f(b')))$. For details and references, see [HM1,§2].

Notice that in [HM1], bracket notation is used for Lie algebroid actions, so that $X^*(u')$ is denoted there by $[X, u']$.

It is worth observing that this correspondence between action morphisms and actions of Lie algebroids illustrates a process which we will use again in what follows, namely: (i) Characterize a groupoid (or double groupoid) construction in terms of a class of groupoid morphisms or maps; (ii) Apply the Lie functor to this class and characterize the resulting Lie algebroid morphisms abstractly, without reference to the differentiation process by which they were obtained; (iii) Prove that this class of Lie algebroid morphisms corresponds to a Lie algebroid construction analogous to the original groupoid construction. This is the method used throughout [HM1].

We now return to $\mathcal{LA}$-groupoids. In detail, an $\mathcal{LA}$-groupoid comprises the data in Figure 7 in which $G$ is a Lie groupoid on $B$ and $A$ is a Lie algebroid on $B$, and in which $\Omega$ has both

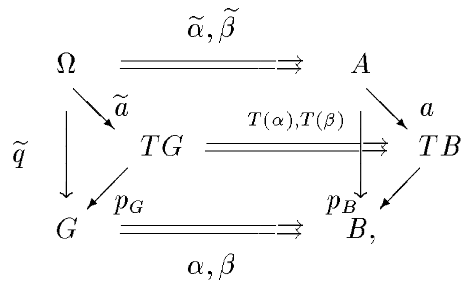



Figure 7.

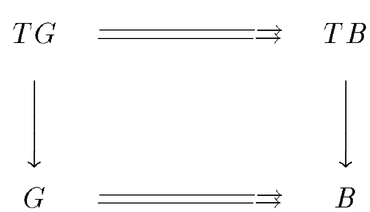
a Lie algebroid structure on base $G$ and a Lie groupoid structure on base $A$, such that the structure maps of the Lie groupoid structure on $\Omega$ are Lie algebroid morphisms and, finally, such that the double source map $(\widetilde{q}, \widetilde{\alpha}): \Omega \to G \underset{B}{\times} A$ is a surjective submersion.

We denote the two anchors by $a: A \to TB$ and $\widetilde{a}: \Omega \to TG$, and the two brackets by the one symbol $[\ ,\ ]$.

**Example 4.2** (Compare [CDW], [P4].) Let $G$ be a Lie groupoid on $B$. Then, because the tangent functor preserves pullbacks, $TG$ has a differentiable groupoid structure on $TB$, with source $T(\alpha)$, target $T(\beta)$, identities $T(1)(X)$ for $X \in TB$, and multiplication

$$TG \underset{TB}{\times} TG \cong T(G \underset{B}{\times} G) \xrightarrow{\ T(\kappa)\ } TG$$

where $\kappa$ is the multiplication in $G$. All the maps of this groupoid structure are morphisms of vector bundles and—since they are all tangent maps—of Lie algebroids. Thus




is an $\mathcal{LA}$-groupoid (both of whose anchors are identities), the condition on the double source map following from the fact that $\alpha: G \to B$ is a surjective submersion. We denote the groupoid operation in $TG$ by $X \bullet Y$, where $T(\alpha)(X) = T(\beta)(Y)$.

If $G$ is a Lie group then there is a simple explicit formula for the group operation in $TG$. Namely, for $X \in T(G)_g$, $Y \in T(G)_h$,

$$X \bullet Y = T(L_g)_h(Y) + T(R_h)_g(X) \tag{13}$$

where $L_g$, $R_h$ are the left- and right-translations in $G$. The identity is $0_1 \in T(G)_1$ and the inverse of $X \in T(G)_g$ is $-T(L_{g^{-1}}) \circ T(R_{g^{-1}})(X)$. Such formulas cannot be given for general groupoids. They reflect the fact that, for $G$ a Lie group, $TG \cong G \ltimes \mathcal{G}$, the (group) semi-direct product of $G$ with its Lie algebra by the adjoint representation. ∎

The concept of $\mathcal{LA}$-groupoid lacks the symmetry of that of double groupoid or double vector bundle, since it is not clear what could be meant by requiring a Lie algebroid bracket to be a morphism of Lie groupoids. We therefore set out the conditions in detail. First of all, an $\mathcal{LA}$-groupoid $(\Omega; G, A; B)$ has an underlying $\mathcal{VB}$-groupoid structure, and the anchors $\widetilde{a}\colon \Omega \to TG$, $a\colon A \to TB$, together with the identity maps on $B$ and $G$, form a morphism of $\mathcal{VB}$-groupoids $(\Omega; G, A; B) \to (TG; G, TB; B)$, and indeed a morphism of $\mathcal{LA}$-groupoids. The remaining requirements are that the groupoid structure maps $\widetilde{\alpha}\colon \Omega \to A$, $\widetilde{\beta}\colon \Omega \to A$, $\widetilde{1}\colon A \to \Omega$ and $\widetilde{\delta}\colon \Omega \underset{\alpha}{\times} \Omega \to \Omega$ preserve the brackets of the Lie algebroid structures.

Applying 4.1 to $\widetilde{\alpha}\colon \Omega \to A$, $\alpha\colon G \to B$, we say that $\xi \in \Gamma_G\Omega$ is *$\alpha$-projectable* (rather than $\widetilde{\alpha}$-projectable) if there exists $X \in \Gamma A$ such that $\widetilde{\alpha} \circ \xi = X \circ \alpha$, and we see that the bracket condition on $\widetilde{\alpha}$ is equivalent to

$$\left\{ \begin{array}{c} \text{If } \xi, \eta \in \Gamma_G\Omega \text{ are } \alpha\text{-projectable, with } \xi \overset{\alpha}{\sim} X \in \Gamma A,\ \eta \overset{\alpha}{\sim} Y \in \Gamma A, \\ \text{then } [\xi, \eta] \text{ is } \alpha\text{-projectable, and } [\xi, \eta] \overset{\alpha}{\sim} [X, Y]. \end{array} \right\} \tag{14}$$

Similar remarks apply to $\widetilde{\beta}\colon \Omega \to A$. For $\widetilde{\delta}$, note first that the domain of $\widetilde{\delta}$ is the pullback

$$\begin{array}{ccc} \Omega \underset{\alpha}{\times} \Omega & \longrightarrow & \Omega \\ \big\downarrow & & \big\downarrow \widetilde{\alpha} \\ \Omega & \underset{\widetilde{\alpha}}{\longrightarrow} & A \end{array}$$

in the category of Lie algebroids. For the details of the bracket structure on $\Omega \underset{\alpha}{\times} \Omega$, see [HM1, §1]. If $\xi \in \Gamma_G\Omega$ is $\alpha$-projectable, then $\xi \underset{\alpha}{\times} \xi$ will denote the map $(h, g) \mapsto (\xi(h), \xi(g))$, which is a section of the pullback Lie algebroid $\Omega \underset{\alpha}{\times} \Omega$ on $G \underset{\alpha}{\times} G$. As with $\alpha$, we simplify "$\widetilde{\delta}$-projectable" to "$\delta$-projectable".

**Proposition 4.3.** *Let $(\Omega; G, A; B)$ be an $\mathcal{LA}$-groupoid.*

- *(i) Let $\xi \in \Gamma_G\Omega$ be $\alpha$-projectable, with $\xi \overset{\alpha}{\sim} X \in \Gamma A$. Then $\xi \underset{\alpha}{\times} \xi$ is $\delta$-projectable with $\xi \underset{\alpha}{\times} \xi \overset{\alpha}{\sim} \xi$ if and only if $\xi\colon G \to \Omega$ is a morphism of Lie groupoids over $X\colon B \to A$.*
- *(ii) If $\xi, \eta \in \Gamma_G\Omega$ are morphisms over $X, Y \in \Gamma A$, then $[\xi, \eta]$ is a morphism over $[X, Y]$.*

PROOF: If $\xi$ is a morphism over $X$, then $\widetilde{\delta}(\xi(h), \xi(g)) = \xi(h)\xi(g)^{-1} = \xi(hg^{-1}) = \xi(\delta(h, g))$ for all $(h, g) \in G \underset{\alpha}{\times} G$. Conversely, suppose that $\xi \underset{\alpha}{\times} \xi \overset{\delta}{\sim} \xi$. Then first one has $\xi(1_b) = \xi(1_b)\xi(1_b)^{-1}$ so $\xi(1_b) = \widetilde{1}_{X(b)}$ for all $b \in B$. Next it follows that $\widetilde{\beta}(\xi(g)) = \widetilde{\beta}(\xi(g)\xi(g)^{-1}) = X(\beta(g))$ for all $g \in G$. That $\xi$ is a morphism over $X$ now follows.

Now (ii) follows from (i) and 4.1. ■

Condition 4.3(ii) is often of interest in its own right. That it is weaker than the interchange law can be seen from 4.2: there a morphic section is a pair of vector fields $\xi \in \Gamma TG$, $X \in \Gamma TB$ such that $T(\alpha)\circ\xi = X\circ\alpha$, $T(\beta)\circ\xi = X\circ\beta$ and $\xi(hg) = \xi(h)\bullet\xi(g)$ whenever $g, h \in G$ are composable; in particular, $\xi(1_b) = T(1)(X(b))$ for all $b \in B$, and so the morphic vector fields cannot generate $\Gamma TG$ over $C(G)$.

We turn now to the $\mathcal{LA}$-groupoids attached to a double Lie groupoid

$$\begin{array}{ccc} S & \overset{\widetilde{\alpha}_H,\,\widetilde{\beta}_H}{\Longrightarrow} & V \\ {\scriptstyle \widetilde{\alpha}_V,\,\widetilde{\beta}_V}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \alpha_V,\,\beta_V} \\ H & \underset{\alpha_H,\,\beta_H}{\Longrightarrow} & B \end{array} .$$

Denote the Lie algebroid of the vertical groupoid structure on $S$ by $A_VS$, with bundle projection $\widetilde{q}_V\colon A_VS \to H$ and anchor $\widetilde{a}_V\colon A_VS \to TH$. Denote the bundle projection of $AV$ by $q_V\colon AV \to B$ and its anchor by $a_V\colon AV \to TB$. We claim that Figure 8 is an $\mathcal{LA}$-groupoid.

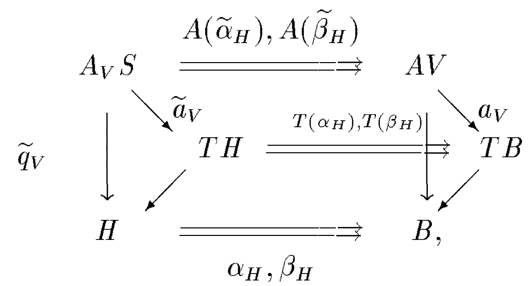



Figure 8.

Here, to obtain the groupoid operations in $A_VS$, one regards the horizontal division in $S$ as a morphism of groupoids

$$\begin{array}{ccc} S_V \underset{V}{\times} S_V & \overset{\widetilde{\delta}_H}{\longrightarrow} & S_V \\ \Big\Downarrow & & \Big\Downarrow \\ H \underset{\alpha}{\times} H & \underset{\delta_H}{\longrightarrow} & H \end{array}$$

where $S_V \underset{V}{\times} S_V$ is the groupoid pullback

$$\begin{array}{ccc} S_V \underset{V}{\times} S_V & \longrightarrow & S_V \\ \Big\downarrow & & \Big\downarrow \; \widetilde{\alpha}_H \\ S_V & \xrightarrow[\widetilde{\alpha}_H]{} & V \end{array}$$

and applies the Lie functor $A$.

That this yields a groupoid structure follows easily from the facts that the groupoid axioms can be expressed diagrammatically, and that the Lie functor $A$ preserves pullbacks. We call $(A_V S; H, AV; B)$ the *vertical* $\mathcal{LA}$*-groupoid* of $(S; H, V; B)$. There is clearly also a *horizontal* $\mathcal{LA}$*-groupoid*, denoted $(A_H S; AH, V; B)$, with diagram as in Figure 9.

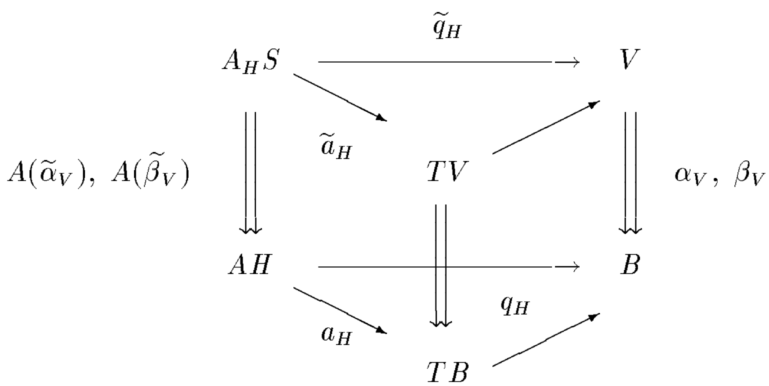



Figure 9.

**Example 4.4** For $G$ a Lie groupoid on $B$, the double Lie groupoid

$$\begin{array}{ccc} G \times G & \rightrightarrows & G \\ \downdownarrows & & \downdownarrows \\ B \times B & \rightrightarrows & B \end{array}$$

of 2.3 has vertical $\mathcal{LA}$-groupoid as in Figure 10 where the Lie algebroid structure on $AG \times AG$ is the Cartesian square of $AG$, with base $B \times B$, and the groupoid structure is the pair groupoid structure with base $AG$. The horizontal $\mathcal{LA}$-groupoid structure is as in Figure 11; compare 4.2. For any Lie algebroid $A$ on $B$, there is an evident $\mathcal{LA}$-groupoid, as in Figure 12, which abstracts the first of these. ■

**Example 4.5** Let $H$ and $V$ be Lie groupoids on a common base $B$, and let $S$ be the double

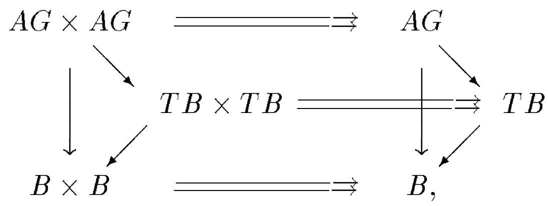



Figure 10.

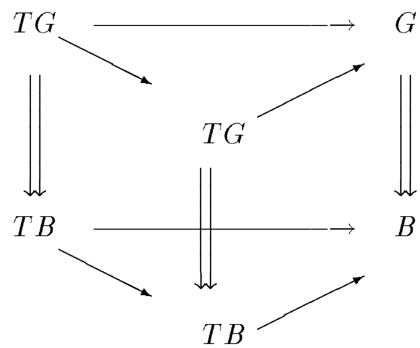



Figure 11.

groupoid $\square(H, V)$ of 2.4; assume that the anchors of $H$ and $V$ are suitably transversal. Then the vertical groupoid structure $S_V \Longrightarrow H$ is the pullback Lie groupoid $\chi_H^{**}(V \times V)$ and therefore has for Lie algebroid the pullback

$$\begin{array}{ccc} \chi_H^{**}(AV \times AV) & \longrightarrow & AV \times AV \\ \downarrow & & \downarrow \quad a_V \times a_V \\ TH & \xrightarrow[T(\chi_H)]{} & TB \times TB \end{array} \qquad (15)$$

We denote elements of $\chi_H^{**}(AV \times AV)$ by $(Z, Y, X)$, where $Z \in T(H)_h, Y \in AV|_y, X \in AV|_x$ and $\beta(h) = y, \alpha(h) = x$, $T(\beta)(Z) = Y$, $T(\alpha)(Z) = X$. Now the groupoid structure on $\chi_H^{**}(AV \times AV)$ may easily be seen to have source and target maps $A(\widetilde{\alpha}_H)\colon (Z, Y, X) \mapsto X$ and $A(\widetilde{\beta}_H)\colon (Z, Y, X) \mapsto Y$, and multiplication $(Z', Y', X')(Z, Y, X) = (Z' \bullet Z, Y', X)$, defined iff $X' = Y$. (Here $\bullet$ is the groupoid operation in $TH$.) Thus the groupoid structure on $\chi_H^{**}(AV \times AV)$ is precisely that obtained by regarding (15) as the pullback of Lie groupoids

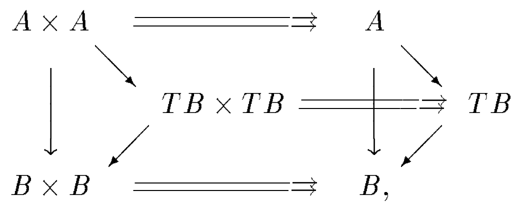



Figure 12.

$$\begin{array}{ccc} a_V^{**}(TH) & \longrightarrow & TH \\ \Big\downarrow & & \Big\downarrow \qquad T(\chi_H) = \chi_{TH} \\ AV \times AV & \underset{a_V \times a_V \,=\, a_{V \times V}}{\longrightarrow} & TB \times TB. \end{array}$$

We represent these structures by the diagram in Figure 13 where the symbol $\approx$ may be

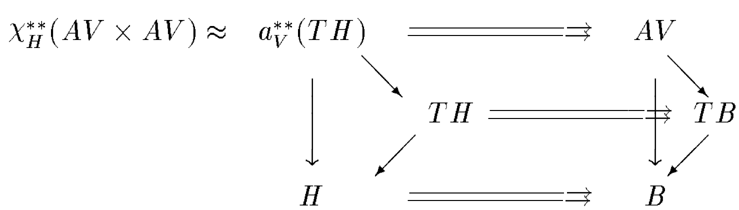



Figure 13.

regarded as denoting diffeomorphism of the underlying manifolds, or isomorphism of understood $\mathcal{LA}$-groupoid structures.

This $\mathcal{LA}$-groupoid structure may be given without presuming the presence of the groupoid $V$. Suppose given a locally trivial Lie groupoid $G$ and any transitive Lie algebroid $A$ on the one base $B$. Then the pullback of

$$\begin{array}{ccc} & & A \times A \\ & & \Big\downarrow \quad a \times a \\ TG & \underset{T(\chi)}{\longrightarrow} & TB \times TB \end{array}$$

may be regarded as either the pullback groupoid $a^{**}(TG)$ or the pullback Lie algebroid $\chi^{**}(A\times A)$, and these two structures constitute an $\mathcal{LA}$-groupoid. (As with all such examples, a much weaker transitivity condition suffices.) ■

The groupoid structure on $A_V S$ in 4.5 may also be calculated by means of the following lemma.

**Lemma 4.6.** *If $(\varphi;\varphi_H,\varphi_V;\varphi_B)\colon (S';H',V';B') \to (S;H,V;B)$ is a morphism of double Lie groupoids, and if $\varphi\colon S'_H \to S_H$, $\varphi_V\colon V' \to V$ is a fibration, regular fibration, action morphism or inductor, then $A_V(\varphi)\colon A_V S' \to A_V S$, $A(\varphi_V)\colon AV' \to AV$ is also a fibration, regular fibration, action morphism or inductor of Lie groupoids, respectively.* ■

This may be proved by observing that the tangent functor preserves pullbacks, and that the Lie algebroid of a Lie groupoid is the restriction of a tangent bundle to the manifold of identity elements.

We write $A_V(\varphi)$ in 4.6 to emphasize that the Lie functor is being applied to the vertical structure. Returning to 4.5, we know that $\widetilde{\chi}_V\colon S_H \to H\times H$, $\chi_V\colon V \to B\times B$ is an inductor, since $S_H$ is the pullback groupoid $\chi_V^{**}(H\times H)$. From 4.6 it then follows that $\widetilde{a}_V = A(\widetilde{\chi}_V)\colon A_V S \to TH$, $a_V = A(\chi_V)\colon AV \to TB$ is an inductor of Lie groupoids, and so the groupoid structure on $A_V S$ is $a_V^{**}(TH)$.

**Example 4.7** Let $H$ and $V$ be Lie groupoids on the one base $B$, and let $\varphi\colon H \to V$ be a base-preserving morphism. Let $\Theta = \Theta(H,\varphi,V)$ be the comma double groupoid of 2.5. Then the vertical structure of $\Theta$ is the pullback groupoid $\alpha_H^{**}(V)$ on base $H$, and so $A_V\Theta$ is the pullback Lie algebroid

$$\begin{array}{ccc}
\alpha_H^{**}(AV) & \longrightarrow & AV \\
\Big\downarrow & & \Big\downarrow a_V \\
TH & \xrightarrow[T(\alpha_H)]{} & TB.
\end{array}$$

We denote elements of $\alpha_H^{**}(AV)$ by $(Z,X)$ where $Z\in T(H)_h, X\in AV|_x, \alpha(h)=x$, and $T(\alpha_H)(Z) = a_V(X)$. Now $\widetilde{\chi}_V\colon \Theta_H \to H\times H$, $\chi_V\colon V\to B\times B$ is an action morphism (since $\Theta_H \cong (H\times H)\ltimes \chi_V$) so, applying 4.6, $\widetilde{a}_V\colon A_V\Theta \to TH$, $a_V\colon AV \to TB$ is also an action morphism of Lie groupoids, and so $A_V\Theta \cong TH\ltimes a_V$, with respect to an induced action of the groupoid $TH \rightrightarrows TB$ on $a_V\colon AV\to TB$. This action is obtained by applying the Lie functor to the action of $H\times H$ on $\chi_V$, namely $(h_2,h_1)(v) = \varphi(h_2)v\varphi(h_1)^{-1}$, and this is

$$Z\bullet X = T(R_{h^{-1}})(T(\kappa_V)(T(\varphi)(Z),V))$$

where $Z$ and $X$ are as above. Here $\kappa_V\colon V*V \to V$ is the multiplication in $V$. Then the source $A(\widetilde{\alpha}_H)$ maps $(Z,X)$ to $X$, the target $A(\widetilde{\beta}_H)$ maps $(Z,X)$ to $Z\bullet X$, and the groupoid composition in $A_V\Theta$ is

$$(Z',X')(Z,X) = (Z'\bullet Z, X),$$

defined iff $X' = Z \bullet X$. Here $\bullet$ denotes both the action of $TH$ on $a_V$ and the multiplication in $TH$. We display the $\mathcal{LA}$-groupoid $A_V\Theta$ as in Figure 14.

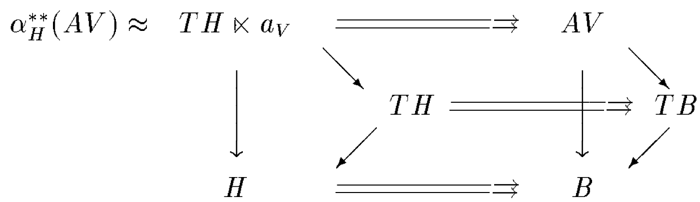



Figure 14.

Proceeding now to obtain $A_H\Theta$, the horizontal structure of $\Theta$ is the action groupoid $(H \times H) \ltimes \chi_V$, where the Cartesian product groupoid $H \times H$ acts on $\chi_V: V \to B \times B$ by $(h_2, h_1)(v) = \varphi(h_2)v\varphi(h_1)^{-1}$. Thus the Lie algebroid $A_H\Theta$ is $(AH \times AH) \ltimes \chi_V$, and we denote the elements of this by $(Y, X, v)$ where $Y \in AH, X \in AH, v \in V$ and $\beta(v) = q_H(Y), \alpha(v) = q_H(X)$. It is straightforward to verify that the action of the Lie algebroid $AH \times AH$ on $\chi_V$ is

$$(Y,X)^*(v) = \overrightarrow{A(\varphi)(Y)}(v) + T(\sigma)(\overrightarrow{A(\varphi)(X)}(v^{-1}))$$

for $Y, X, v$ as above (where we are using the notation for actions described at the start of the section, rather than that of [HM1]). Here the arrows denote the right-translations of elements of $AV$, as in [M1, III§3], and $\sigma$ is the inversion $v \mapsto v^{-1}$ of $V$.

Again applying 4.6, the groupoid structure of $A_H\Theta$ is the pullback $q_H^{**}(V)$, with source and target maps $A(\widetilde{\alpha}_V): (Y, X, v) \mapsto X$ and $A(\widetilde{\beta}_V): (Y, X, v) \mapsto Y$, and multiplication

$$(Y', X', v')(Y, X, v) = (Y', X, v'v),$$

defined iff $X' = Y$. Here $q_H$ is the bundle projection $AH \to B$. Diagrammatically, we have Figure 15.

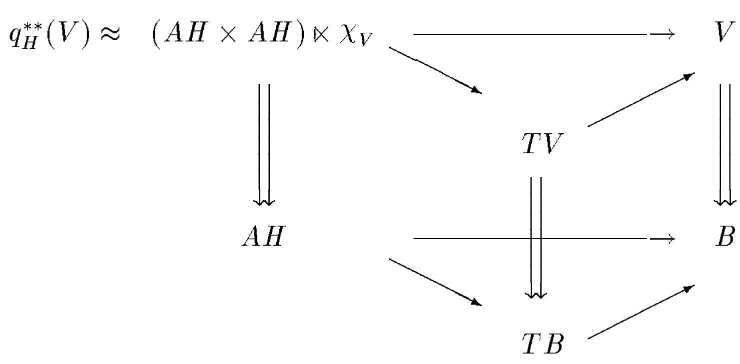



Figure 15.

It is possible to give a version of this $\mathcal{LA}$-groupoid also without presuming the presence of the groupoid $V$. Let $A$ and $G$ be a Lie algebroid and a Lie groupoid on the same base $B$, and let $\varphi\colon A \to AG$ be a morphism of Lie algebroids over $A$. Then the diagram

$$\begin{array}{ccc} & & G \\ & & \downarrow\ \chi \\ A \times A & \xrightarrow[q \times q]{\qquad} & B \times B \end{array}$$

defines both the pullback groupoid $q^{**}(G)$ and the action Lie algebroid $(A \times A) \ltimes \chi$, where the action is

$$(Y, X)^*(g) = \overrightarrow{\varphi(Y)}(g) + T(\sigma)(\overrightarrow{\varphi(X)}(g^{-1})),$$

the notation being as before. It is routine to verify that these two structures make $q^{**}(G) \approx (A \times A) \ltimes \chi$ an $\mathcal{LA}$-groupoid.

It is not clear whether an abstract version of $A_V\Theta$ can be given. ∎

**Example 4.8** The compatibility conditions on an $\mathcal{LA}$-groupoid simplify somewhat when the double base is a point. Let $G$ be a Lie group and $A$ a Lie algebra. Then an $\mathcal{LA}$-groupoid $(\Omega; G, A; *)$ involves both a Lie group*oid* structure on $\Omega$ with base $A$, and a Lie algebr*oid* structure with base $G$.

Consider $\widetilde{\alpha}\colon \Omega \to A$, which is to be a morphism of Lie algebroids over $G \to *$. The anchor condition is vacuous. A section $\xi \in \Gamma_G\Omega$ is $\alpha$-projectable to $X \in A$ if and only if $\xi$ takes values in $\Omega_X = \widetilde{\alpha}^{-1}(X)$. Thus $\widetilde{\alpha}$ is a Lie algebroid morphism if and only if the bracket $[\xi, \eta]$ of sections $\xi\colon G \to \Omega_X$ and $\eta\colon G \to \Omega_Y$, where $X, Y \in A$, takes values in $\Omega_{[X,Y]}$. The condition on $\widetilde{\beta}$ is similar.

That multiplication $\Omega * \Omega \to \Omega$ commutes with the anchors becomes

$$\widetilde{a}(\eta\xi) = T(L_h)(\widetilde{a}(\xi)) + T(R_g)(\widetilde{a}(\eta))$$

for $\eta, \xi \in \Omega$ with $\widetilde{q}(\eta) = h, \widetilde{q}(\xi) = g$. Sections of the pullback Lie algebroid $\Omega * \Omega \to G \times G$ may be written as $\eta * \xi$, where $\eta \in \pi_l^*\Omega$, $\xi \in \pi_r^*\Omega$ are sections of the pullbacks of $\Omega \to G$ over the projections $\pi_l$, $\pi_r\colon G \times G \to G$, and $\widetilde{\alpha} \circ \eta = \widetilde{\beta} \circ \xi$. Suppose $\eta * \xi \stackrel{\kappa}{\sim} \zeta$; that is, $\eta(h, g)\xi(h, g) = \zeta(hg)$ for all $h, g \in G$. Similarly suppose that $\eta' * \xi' \in \Gamma_{G\times G}(\Omega * \Omega)$ and $\eta' * \xi' \stackrel{\kappa}{\sim} \zeta'$. Then the interchange law in $\Omega$ states that

$$[\eta * \xi, \eta' * \xi'] \stackrel{\kappa}{\sim} [\zeta, \zeta'].$$

A morphic section $\xi \in \Gamma_G\Omega$ is simply a section taking values in $\Omega_X^X = \widetilde{\alpha}^{-1}(X) \cap \widetilde{\beta}^{-1}(X)$ for some fixed $X \in A$, and which is a morphism of Lie groups $G \to \Omega_X^X$. Thus 4.3(ii) asserts that if $\xi\colon G \to \Omega_X^X$ and $\eta\colon G \to \Omega_Y^Y$ are morphic sections then $[\xi, \eta]\colon G \to \Omega_{[X,Y]}^{[X,Y]}$ is also.

We refer to an $\mathcal{LA}$-groupoid whose double base is singleton as an $\mathcal{LA}$*-group*. If an $\mathcal{LA}$-groupoid $(\Omega; G, A; B)$ on a general base $B$ satisfies suitable local triviality conditions, then at each $b \in B$ there will be an $\mathcal{LA}$-group with double base $\{b\}$. Descriptions of $\mathcal{LA}$-groupoids in these terms are subsumed under §5. ∎

We turn now to the $\mathcal{LA}$-groupoids of vacant double groupoids. Consider an interacting pair of Lie groupoids $H$ and $V$, as in 2.14, and write $S = H \bowtie V$. The horizontal groupoid $S_H \rightrightarrows V$ is then the action groupoid $H \ltimes \beta_V$ and so the Lie algebroid $A_H S$ is $AH \ltimes \beta_V$. As a manifold, this is the pullback $AH \underset{B}{\times} V$ of $q_H\colon AH \to B$ and $\beta_V$, and we denote elements of it by $(X, v)$. The action of $H$ on $\beta_V$ induces an action of $AH$ on $\beta_V$. Namely, given $X \in AH$, $v \in V$ with $q(X) = \beta(v)$, define ${}^X v = X^*(v) \in T(V)_v$ by

$$ {}^X v = T(h \mapsto {}^h v)_{1^H_{\beta(v)}}(X(\beta(v)). $$

The anchor $\widetilde{a}_H\colon AH \ltimes \beta_V \to TV$ is now $(X, v) \mapsto {}^X v$.

In this case the groupoid structure on $A_H S$ is more easily determined without 4.6. The target map is evidently $A(\widetilde{\beta}_V)\colon (X, v) \mapsto X$. To describe the source map, first define a right action of $V$ on $q_H\colon AH \to B$ by

$$ X^v = T(h \mapsto h^v)_{1^H_{\beta(v)}}(X) $$

where $q_H(X) = \beta_V(v)$. This is a linear action of $V$ on the vector bundle $AH$. Then the source map $A(\widetilde{\alpha}_V)\colon AH \ltimes \beta_V \to AH$ is clearly $(X, v) \mapsto X^v$ and the groupoid multiplication is

$$ (X_2, v_2)(X_1, v_1) = (X_2, v_2 v_1), $$

defined iff $X_2^{v_2} = X_1$. Thus the groupoid structure of $A_H S$ is that of the right action groupoid $q_H \rtimes V$, and the $\mathcal{LA}$-groupoid $A_H S$ is described by the diagram in Figure 16.

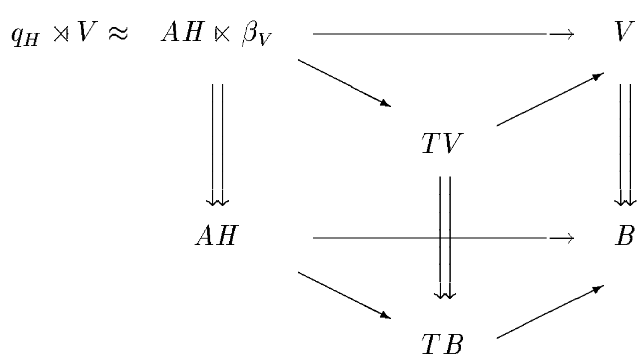



Figure 16.

The five compatibility conditions of 2.10 differentiate to give compatibility conditions between the left action of $AH$ on $\beta_V$ and the right action of $V$ on $q_H$. Firstly, (i) of 2.10 implies that

$$ T(\alpha_V)({}^X v) = a_H(X^v) \tag{16} $$

for $X \in AH$, $v \in V$ with $q_H(X) = \beta_V(v)$. This is easy to verify. Next, (ii) implies that

$$ {}^X(1^V_b) = T(1^V)(a_H(X)(b)) \tag{17} $$

for $X \in AH|_b$, $b \in B$, and (iv) implies that

$$^{X}(v_2 v_1) = {}^{X}v_2 \bullet {}^{X^{v_2}}v_1 \tag{18}$$

for $X \in AH$ and $v_2$, $v_1 \in V$ with $q_H(X) = \beta(v_2)$ and $\alpha(v_2) = \beta(v_1)$. Equation (iii) disappears into the definition of the action of $V$ on $q_H$.

Equation (v) is the condition that $\widetilde{\alpha}_V: S_H \to H$, $\alpha_V: V \to B$ is a morphism of differentiable groupoids. Its differentiated form is therefore the statement that $A(\widetilde{\alpha}_V): A_H S \to AH$, $(X, v) \mapsto X^v$, is a morphism of Lie algebroids over $\alpha_V$. Since $\alpha_V$ is a surjective submersion, and since (16) asserts that $A(\widetilde{\alpha}_V)$ preserves the anchors, it follows from 4.1 that this is equivalent to the bracket condition (14). To express this more clearly, notice that given $X \in \Gamma AH$ and $\xi \in \Gamma_V(AH \ltimes \beta_V)$, and writing $\xi(v) = (\bar{\xi}(v), v)$ where $\bar{\xi}(v) \in AH|_{\beta(v)}$ for all $v \in V$, we have $\xi \overset{\alpha}{\sim} X$ if and only if $\bar{\xi}(v)^v = X(\alpha v)$ for all $v \in V$. Now for any $X \in \Gamma AH$, define $\vec{X} \in \Gamma_V(AH \ltimes \beta_V)$ by

$$\vec{X}(v) = \left(X(\alpha v)^{v^{-1}}, v\right)$$

for $v \in V$. Then $\xi \overset{\alpha}{\sim} X$ if and only if $\xi = \vec{X}$. The bracket-preservation property now takes the simple form

$$\overrightarrow{[X, Y]} = [\vec{X}, \vec{Y}] \tag{19}$$

and this is the differentiated form of (v) of 2.10.

The equations (i)—(v) of 2.10 do not include a condition corresponding to the interchange law in the double groupoid, for this follows automatically since elements of $H \bowtie V$ are determined by their sides. It is nonetheless interesting to notice the differentiated form of 4.3(ii). Continuing the above notation, $\xi \in \Gamma_V(AH \ltimes \beta_V)$ is a morphism over $X \in \Gamma AH$ if and only if it commutes with both the source and the target projections; that is, if and only if $\bar{\xi}(v) = X(\beta v)$ and $\bar{\xi}(v)^v = X(\alpha v)$ for all $v \in V$. These are equivalent to the single equation $X(\beta v)^v = X(\alpha v)$, $v \in V$, and we say that $X \in \Gamma AH$ is $V$-*equivariant* if this equation holds. The interchange law in $H \bowtie V$ therefore implies that

$$\text{If } X, Y \in \Gamma AH \text{ are } V-\text{equivariant, then } [X, Y] \text{ is also.} \tag{20}$$

In fact, (19) implies (20). For any $X \in \Gamma AH$ define $\overline{X} \in \Gamma_V(AH \ltimes \beta_V)$ by $\overline{X}(v) = (X(\beta v), v)$. Then $\overline{X} \overset{\beta}{\sim} X$ and since $\widetilde{\beta}: AH \ltimes \beta_V \to AH$, $(X, v) \mapsto X$ is automatically a Lie algebroid morphism (being the action morphism for $AH \ltimes \beta_V$), it follows that $[\overline{X}, \overline{Y}] = \overline{[X, Y]}$. Now $X \in \Gamma AH$ is $V$-equivariant if and only if $\vec{X} = \overline{X}$. So it follows that if $X$, $Y \in \Gamma AH$ are $V$-equivariant, then

$$\overrightarrow{[X, Y]} = [\vec{X}, \vec{Y}] = [\overline{X}, \overline{Y}] = \overline{[X, Y]},$$

and therefore $[X, Y]$ is $V$-equivariant.

Now suppose $(\Omega; G, A; B)$ is an $\mathcal{LA}$-groupoid for which the double source map $(\widetilde{q}, \widetilde{\alpha}): \Omega \to G \underset{B}{\times} A$ is a diffeomorphism. It will be convenient to denote by $(X|g)$, for $X \in A$ and $g \in G$ with $q(X) = \beta(g)$, the element of $\Omega$ with $\widetilde{q}(X|g) = g$ and $\widetilde{\beta}(X|g) = X$.

The double source condition ensures that $\widetilde{q}: \Omega \to G$, $q: A \to B$ is an action morphism of Lie groupoids. The corresponding right action of $G$ on $q: A \to B$ is

$$X^g = \widetilde{\beta}(\widetilde{q}^{*^{-1}}(X, g^{-1})),$$

where $qX = \beta g$. It follows that $\widetilde{\alpha}(X|g) = X^g$, for if $\xi = \widetilde{q}^{*^{-1}}(X, g^{-1})$, then $\widetilde{q}(\xi) = g^{-1}$ and $\widetilde{\alpha}(\xi) = X$, so $\widetilde{q}(\xi^{-1}) = g$ and $\widetilde{\beta}(\xi^{-1}) = X$, whence $\xi^{-1} = (X|g)$ and so $\widetilde{\alpha}(X|g) = \widetilde{\beta}(\xi) = X^g$.

Next, $\widetilde{\beta}: \Omega \to A$, $\beta: G \to B$ is an action morphism of Lie algebroids, again by the double source condition. Given $X \in \Gamma A$, denote by $\overline{X}$ the unique section in $\Gamma_G \Omega$ which projects under $\widetilde{\beta}$ to $X$; clearly $\overline{X}(g) = (X|g)$. Then the corresponding action of $A$ on $\beta: G \to B$ is given, in the usual notation, by $X^* = \widetilde{a}(\overline{X}) \in \Gamma TG$; we will also write ${}^X g = X^*(g) = \widetilde{a}(X|g)$.

Further, $\widetilde{\alpha}: \Omega \to A$, $\alpha: G \to B$ is a morphism of Lie algebroids (indeed an action morphism). For $X \in \Gamma A$, the section, denoted $\vec{X}$, in $\Gamma_G \Omega$ which projects under $\widetilde{\alpha}$ to $X$ is given by $\vec{X}(g) = ((X(\alpha g)^{g^{-1}}|g)$.

We can now deduce analogues of (16)–(19). Since $\widetilde{\alpha}$ is a morphism of Lie algebroids, we have $[\vec{X}, \vec{Y}] = \overrightarrow{[X, Y]}$, as in (19). Since $\widetilde{a}: \Omega \to TG$, $a: A \to TB$ is a morphism of Lie groupoids, we have $\widetilde{a}(\xi\eta) = \widetilde{a}(\xi) \bullet \widetilde{a}(\eta)$ for $\xi, \eta \in \Omega$ and, in particular, taking $\xi = (X|g_2)$, $\eta = (X^{g_2}|g_1)$, so that $\xi\eta = (X|g_2 g_1)$, we have ${}^X(g_2 g_1) = {}^X(g_2) \bullet {}^{X^{g_2}}(g_1)$, as in (18). Returning to $\widetilde{\alpha}: \Omega \to A$, the anchor condition $T(\alpha) \circ \widetilde{a} = a \circ \widetilde{\alpha}$ applied to $(X|g)$ gives $T(\alpha)({}^X g) = a(X^g)$, as in (16). Lastly, (17) follows from (18).

**Theorem 4.9.** *Let $A$ be a Lie algebroid, and let $G$ be a Lie groupoid on the same base $B$. Suppose given a left action $(X, g) \mapsto {}^X g$, $A \underset{B}{\times} G \to TG$ of $A$ on $\beta: G \to B$, and a right linear action $(X, g) \mapsto X^g$, $A \underset{B}{\times} G \to A$ of $G$ on $q: A \to B$. Then $A \underset{B}{\times} G$, the pullback of $q$ and $\beta$, supports both the Lie algebroid structure $A \ltimes \beta$ on base $G$, and the Lie groupoid structure $q \rtimes G$ on base $A$, and these constitute an $\mathcal{LA}$-groupoid structure if and only if the actions obey the following four conditions:*

*(i) $T(\alpha)({}^X g) = a(X^g)$ for $X \in A$ and $g \in G$ with $q(X) = \beta(g)$;*

*(ii) ${}^X 1_b = T(1)(a(X))$ for $X \in A_b$, $b \in B$;*

*(iii) ${}^X(g_2 g_1) = {}^X(g_2) \bullet {}^{X^{g_2}}(g_1)$ for $X \in A$ and $g_2, g_1 \in G$ with $q(X) = \beta(g_2)$ and $\alpha(g_2) = \beta(g_1)$;*

*(iv) $\overrightarrow{[X, Y]} = [\vec{X}, \vec{Y}]$ for $X, Y \in \Gamma A$, where $\vec{X} \in \Gamma(A \ltimes \beta)$ is defined by $\vec{X}(g) = \left(X(\alpha g)^{g^{-1}}, g\right)$ for $g \in G$.*

PROOF: Suppose that the conditions are satisfied. The target map $\widetilde{\beta}: A \ltimes \beta \to A$, $(X, g) \mapsto X$, is the action morphism arising from the action of $A$ on $\beta$, and so is automatically a Lie algebroid morphism. The source map $\widetilde{\alpha}: A \ltimes \beta \to A$, $(X, g) \mapsto X^g$ preserves the anchors by (i) and the brackets by (iv), as in the preceding discussion. The identity map $\widetilde{1}: A \to A \ltimes \beta$, $X \mapsto (X, 1_{qX})$ preserves the anchors by (ii) and the brackets automatically.

It remains to verify that the groupoid multiplication $(A \ltimes \beta) * (A \ltimes \beta) \to (A \ltimes \beta)$ is a Lie algebroid morphism. We use the following general device. Consider any Lie algebroid $A$ on base $B$ acting on maps $M \to B$ and $N \to B$ in such a way that a certain base-preserving map $f: M \to N$ is equivariant. That is, if $X \mapsto X^*$ is the action on $M$ and $X \mapsto X^\dagger$ is the action on $N$, we require that $X^* \overset{f}{\sim} X^\dagger$ for all $X \in \Gamma A$. Then the induced map $A \ltimes M \to A \ltimes N$ is readily seen to be a Lie algebroid morphism ([HM1,§2]).

In the present case the action of $A$ on $G$ can be doubled to an action of $A$ on $G * G \to B$, $(g_2, g_1) \mapsto \beta g_2$, by defining

$$^{X}(g_2, g_1) = \left(^{X}(g_2), ^{X^{g_2}}(g_1)\right).$$

That the right hand side of this expression is always tangent to $G * G$ follows from (i), and the other conditions are immediate. From (iii) it follows that multiplication $G * G \to G$ is $A$-equivariant, and so the induced map $A \ltimes (G*G) \to A \ltimes \beta$ is a Lie algebroid morphism. Lastly, the pullback Lie algebroid $(A \ltimes \beta) * (A \ltimes \beta) = \{((Y, g_2), (X, g_1)) \mid Y^{g_2} = X\}$ is isomorphic to $A \ltimes (G * G)$ under the canonical map $((Y, g_2), (X, g_1)) \mapsto (Y, g_2 g_1)$. Clearly the multipication $(A \ltimes \beta) * (A \ltimes \beta) \to A \ltimes \beta$ identifies with the map $A \ltimes (G * G) \to A \ltimes \beta$ induced by the multiplication in $G$.

The converse is a straightforward modification of the argument preceding the theorem. ∎

**Definition 4.10.** *(i) Let $A$ be a Lie algebroid and $G$ be a Lie groupoid on base $B$. Then an* interaction *of $A$ with $G$ is a pair of actions, of $A$ on $\beta: G \to B$ and of $G$ on $q: A \to B$, which satisfy the conditions (i)—(iv) of 4.9. We also then say, briefly, that $A$ and $G$* interact.

*(ii) An $\mathcal{LA}$-groupoid $(\Omega; G, A; B)$ is said to be* vacant *if the double source map $(\widetilde{q}, \widetilde{\alpha}): \Omega \to G \underset{B}{\times} A$ is a diffeomorphism.* ∎

It is worth noting that conditions (iii) and (iv) in 4.9 are dual, in a precise sense. Namely, (iii) is (equivalent to) the condition that $\widetilde{a}: \Omega \to TG$, $a: A \to TB$ is a morphism of Lie groupoids, whilst (iv) is the condition that $\widetilde{\alpha}: \Omega \to A$, $\alpha: G \to B$ is a morphism of Lie algebroids. Condition (iv) cannot easily be replaced by a formula for $[X, Y]^g$ (where $X, Y \in \Gamma A$, $g \in G$), even when the double base is a point.

To summarize, we have the following theorem.

**Theorem 4.11.**

**(i)** *Let $A$ be a Lie algebroid and let $G$ be a Lie groupoid on the same base $B$. Then 4.9 and the discussion preceding it give a bijective correspondence between interactions of $A$ with $G$ and vacant $\mathcal{LA}$-groupoids $(\Omega; G, A; B)$; we denote the vacant $\mathcal{LA}$-groupoid corresponding to a given interaction by $A \bowtie G$, and call it the* indirect product *of $A$ and $G$.*

**(ii)** *The $\mathcal{LA}$-groupoids of a vacant double Lie groupoid are vacant $\mathcal{LA}$-groupoids.*

**(iii)** *If $H$ and $V$ are Lie groupoids on base $B$, and $(h, v) \mapsto {}^{h}v$ and $(h, v) \mapsto h^{v}$ form an interaction of $H$ with $V$, then the actions of $AH$ on $\beta_V$ and of $V$ on $q_H$ arising from these actions form an interaction of $AH$ with $V$. With respect to these interactions, the natural map $A_H(H \bowtie V) \to AH \bowtie V$ is an isomorphism of $\mathcal{LA}$-groupoids preserving $AH$ and $V$.*

∎

One purpose of the concept of vacant $\mathcal{LA}$-groupoid is to explicate the structure of Poisson Lie groups. Recall that a Poisson bracket $\{\ ,\ \}: C(B) \times C(B) \to C(B)$ of smooth functions on a manifold $B$ defines a Lie algebroid structure on the cotangent bundle $T^*B$ with anchor $T^*B \to TB$, $u\,dv \mapsto uX_v$ and bracket

$$[u_1\,dv_1, u_2\,dv_2] = u_1u_2\,d\{v_1, v_2\} + u_1X_{v_1}(u_2)\,dv_2 - u_2X_{v_2}(u_1)\,dv_1,$$

where $X_v$ is the Hamiltonian vector field $X_v(w) = \{v, w\}$ with energy $v \in C(B)$ ([CDW]; see also [Hu] and the references given there). Note that the map $T^*B \oplus T^*B \to B \times \mathbf{R}$ associated to the anchor is the Poisson tensor.

A Poisson bracket on a Lie group $G$ makes $(G, \{\ ,\ \})$ a *Poisson Lie group* if the group multiplication $G \times G \to G$ is a Poisson morphism; that is, maps the induced bracket on $G \times G$ to the given one. Alternatively, the Poisson tensor $T^*G \oplus T^*G \to G \times \mathbf{R}$ must dualize to a morphism of Lie groups $\pi: G \to TG \oplus TG$, where $TG \oplus TG$ is the pullback group, a subgroup of $TG \times TG$. More explicitly, we must have $\pi(hg) = T(L_h)(\pi(g)) + T(R_g)(\pi(h))$ for all $h, g \in G$. Differentiating this equation at the identity gives a map $\mathcal{G} \to \mathcal{G} \oplus \mathcal{G}$ whose dualization is a Lie bracket on $\mathcal{G}^*$. (See [LW1] and references given there.)

For any Lie group $G$, the cotangent bundle has a Lie groupoid structure with base the vector space $\mathcal{G}^*$. The source and target maps are $\widetilde{\alpha}(\theta) = \theta \circ T(L_g)$ and $\widetilde{\beta}(\theta) = \theta \circ T(R_g)$, for $\theta \in T^*(G)_g$, with composition

$$\varphi\theta = \varphi \circ T(R_{g^{-1}}) = \theta \circ T(L_{h^{-1}})$$

for composable $\varphi \in T^*(G)_h$, $\theta \in T^*(G)_g$. Using the trivialization $G \times \mathcal{G}^* \to T^*G$ by right translations, this is precisely the (right) action groupoid $\mathcal{G}^* \rtimes G$ arising from the coadjoint action. ([CDW].) Regarded as a $\mathcal{VB}$-groupoid, $T^*G \Longrightarrow \mathcal{G}^*$ is in a sense dual to the tangent $\mathcal{VB}$-groupoid $(TG; G, \{1\}; \{1\})$ (Pradines [P4]).

**Proposition 4.12.** *Let $(G, \{\ ,\ \})$ be a Poisson Lie group. Then the Lie algebroid structure on $T^*G \to G$ arising from the Poisson bracket, and the groupoid structure $T^*G \Longrightarrow \mathcal{G}^*$ just described make $(T^*G; G, \mathcal{G}^*; 0)$ a vacant $\mathcal{LA}$-groupoid.*

PROOF: The source map of the groupoid structure automatically commutes with the anchors, since the anchor on $\mathcal{G}^*$ is zero. A 1-form $\theta$ will be $\alpha$-projectable to $\omega \in \mathcal{G}^*$ in the sense of 4.1 if and only if $\theta(g) = \omega \circ T(L_{g^{-1}})$ for all $g \in G$; that is, if and only if $\theta$ is left-invariant. That the bracket of left-invariant 1-forms on a Poisson Lie group is left-invariant is known (for example [LW1, 2.1]). Thus $\widetilde{\alpha}: T^*G \to \mathcal{G}^*$ is a morphism of Lie algebroids. Similarly for the target projection.

Sections of the pullback Lie algebroid $T^*G * T^*G \to G \times G$ have the form $(h, g) \mapsto (\theta(g) \circ T(L_{h^{-1}}) \circ T(R_g), \theta(g))$ for a single 1-form $\theta$. Such a section projects to a 1-form $\psi$ under multiplication if and only if $\theta(g) \circ T(L_{h^{-1}}) = \psi(hg)$ for $h, g \in G$, and this can be so if and only if $\psi = \theta$ and $\theta$ is left-invariant. Thus the bracket-preservation property for multiplication also follows from the closure of the left-invariant 1-forms under the Poisson bracket.

The condition that the groupoid multiplication commutes with the anchor simplifies to

$$\widetilde{a}(\varphi \circ T(R_{g^{-1}})) = T(R_g)(\widetilde{a}(\varphi)) + T(L_h)(\widetilde{a}(\varphi \circ T(R_{g^{-1}}) \circ T(L_{h^{-1}})))$$

for $\varphi \in T^*(G)_h$ and $g \in G$. This is equivalent to the twisted multiplicativity condition for the (right) dressing action of $\mathcal{G}^*$ on $G$ ([LW1, 2.4]).

The double source condition and the vacancy condition are clear. ∎

Evidently the action of $G$ on $\mathcal{G}^*$ involved in this structure is the coadjoint action, and the action of $\mathcal{G}^*$ on $G$ is the (right) dressing action. The twisted multiplicativity equation for the dressing action may equally well be regarded as the condition that the anchor is a groupoid morphism from $T^*G \Longrightarrow \mathcal{G}^*$ to the group $TG$, or as (iii) of 4.9. Equation (iv) of 4.9 is in this case the closure of the left-invariant 1-forms under the bracket, and (i) and (ii) are trivial.

Poisson Lie groups are intermediate between Lie bigebras and what might be called "Lie bigroups". That is, associated to every Poisson Lie group $(G, \{\ ,\ \})$ is the Lie bigebra $\mathcal{G} \bowtie \mathcal{G}^*$, and under certain conditions one may integrate $(G, \{\ ,\ \})$ to a bicrossed product $G \bowtie G^*$ (which we refer to as a Lie bigroup). ([Dr], [LW1], [Mj].) The Lie theoretic processes apparent in these constructions have been generalized to a Lie theory for double Lie groups and double Lie algebras in [LW1]. The results of this section show, we believe, that the concept of vacant $\mathcal{LA}$-group is intermediate between that of double Lie group and double Lie algebra, in the same way that Poisson Lie groups are intermediate between Lie bigroups and Lie bigebras.

We hope to give elsewhere a description of general Poisson Lie groupoids ([W1]) in terms similar to 4.12.

# 5 CORE STRUCTURE OF $\mathcal{LA}$-GROUPOIDS

We show that the classification of locally trivial double Lie groupoids by their core structures, as recalled in §2, extends in a natural fashion to $\mathcal{LA}$-groupoids. This result is not automatic, for the constructions on which the double groupoid result depends carry over only in part to Lie algebroids, and it seems unlikely that the corresponding result for double Lie algebroids holds in full. In the second part of the paper we will use this classification in the analysis of connections in double Lie groupoids.

In this section we are mainly concerned with transitive Lie algebroids, and we first recall some results peculiar to this case. Transitivity is the infinitesimal analogue of local triviality: a Lie groupoid $G \Longrightarrow B$ is locally trivial if its anchor $\chi\colon G \to B \times B$ is a surjective submersion, and a Lie algebroid $A$ on $B$ is *transitive* if its anchor $a\colon A \to TB$ is surjective. If a Lie groupoid is locally trivial, then its Lie algebroid is transitive; if the base is connected, the converse is also true ([M1, III 3.16]).

Any locally trivial Lie groupoid is equivalent to the gauge groupoid of a principal bundle ([M1, II §2]), and the Lie algebroid is then precisely the Atiyah sequence of the principal bundle. For a principal bundle $P(B, G, p)$ (with $G$ the structure *group*), this is the exact sequence of vector bundles over $B$,

$$\frac{P \times \mathcal{G}}{G} \overset{j}{\rightarrowtail} \frac{TP}{G} \overset{p_*}{\twoheadrightarrow} TB, \tag{21}$$

where $\frac{TP}{G} \to B$ is the quotient of $TP \to P$ over the action of $G$ and $\frac{P\times\mathcal{G}}{G}$, often denoted $Ad(P)$, is the bundle associated to $P(B, G)$ through the adjoint action of $G$ on its Lie algebra $\mathcal{G}$. The bracket on $\Gamma(\frac{TP}{G})$ is obtained by identifying sections of $\frac{TP}{G}$ with $G$-invariant vector fields on $P$. The map $p_*$ is the quotient of $T(p) : TP \to TB$ and $j$ is induced by the fundamental vector field map $P \times \mathcal{G} \to TP, (u, A) \mapsto A^*(u)$. See [M1, App. A, III 3.20] for further details. Not every transitive Lie algebroid is the Atiyah sequence of a principal bundle ([AMl]).

For a general transitive Lie algebroid $A$ on $B$, we denote the kernel of the anchor by $L$, and call it the *adjoint bundle* of $A$. It is a Lie algebra bundle; that is, the fibres have Lie algebra structures and trivializations exist which preserve these structures ([M1, IV 1.4]).

A *connection* in $A$ is a vector bundle morphism $\gamma: TB \to A$ such that $a \circ \gamma = id_{TB}$, and its *curvature* is the $L$-valued 2-form $R_\gamma$ on $B$ given by $R_\gamma(X, Y) = \gamma[X, Y] - [\gamma X, \gamma Y]$ for $X, Y \in \Gamma TB$; the connection is *flat* if $R_\gamma = 0$. For a Lie algebroid which is the Atiyah sequence (21) of a principal bundle, these notions are precisely the usual ones ([M1, A§4]).

We also need to consider representations of Lie algebroids on vector bundles. For $E$ a vector bundle on $B$, call a first- or zeroth-order differential operator $D: \Gamma E \to \Gamma E$ a *covariant differential operator* if there exists a (necessarily unique) vector field $X$ on $B$ such that $D(f\mu) = fD(\mu) + X(f)\mu$ for $\mu \in \Gamma E$, $f \in C(B)$. The covariant differential operators form the sections of a vector subbundle, denoted $CDO(E)$, of $Diff^1(E, E)$, and this is a Lie algebroid with respect to the bracket $[D, D'] = D \circ D' - D' \circ D$ and anchor $D \mapsto X$, in the above notation. This Lie algebroid is transitive and connections in it are precisely Koszul connections in $E$. Moreover, $CDO(E)$ plays for the vector bundle $E$ the role played for a vector space by its endomorphism Lie algebra. In particular, a *representation* or *action* of a Lie algebroid $A$ with base $B$ on $E$ is a morphism of Lie algebroids $\rho: A \to CDO(E)$ over $B$ ([M1, III§2]). If $A = \frac{TP}{G}$ is the Atiyah sequence of a principal bundle, then a representation of $A$ on $E$ is given by $\widetilde{\rho(X)(\mu)} = \overrightarrow{X}(\widetilde{\mu})$, where $\overrightarrow{X}$ is the $G$-invariant vector field corresponding to $X \in \Gamma(\frac{TP}{G})$, and the tildes denote the $G$-equivariant maps $P \to V$ corresponding to sections of $E$.

The last piece of background that we need concerns Lie algebra bundles. If $L$ is a vector bundle with a tensor field of type $\binom{2}{1}$ which gives a Lie algebra bracket in each fibre of $L$, then $L$ is a Lie algebra bundle if and only if there exists a (Koszul) connection in $L$ such that $\nabla_X([V, W]) = [\nabla_X(V), W] + [V, \nabla_X(W)]$ for all $X \in \Gamma TB$, $V, W \in \Gamma L$ (for example, [M1, III 7.12]). In particular, let $L$ be the kernel of a surjective morphism of transitive Lie algebroids $A \to A'$ over the same base $B$. Take any connection $\gamma$ in $A$, and define $\nabla$ in $L$ by $\nabla_X(V) = [\gamma X, V]$ for $X \in \Gamma TB$, $V \in \Gamma L$. Then $\nabla$ has the required property and so $L$ is a Lie algebra bundle ([M1, IV 1.8]). For any Lie algebra bundle $L$ we denote by $CDO[L]$ the subbundle of $CDO(L)$ whose sections $D$ are derivations of the Lie bracket, that is, $D([V, W]) = [D(V), W] + [V, D(W)]$ for $V, W \in \Gamma L$. This $CDO[L]$ is a transitive Lie subalgebroid of $CDO(L)$ ([M1, p133]).

Until 5.5, consider a fixed $\mathcal{LA}$-groupoid $(\Omega; G, A; B)$. An $\mathcal{LA}$-groupoid has an underlying $\mathcal{VB}$-groupoid structure, and a $\mathcal{VB}$-groupoid is, in particular, a double Lie groupoid. Let $K \rightrightarrows B$ be the core groupoid of this structure; it is easy to see that it is totally intransitive and a vector bundle.

To see that $K$ carries a Lie algebroid structure on $B$, it is helpful to reformulate the definition slightly. Let $\overline{K}$ be the kernel of $\widetilde{\alpha}: \Omega \to A$, regarded as a fibration of Lie algebroids over $\alpha: G \to B$. Then $\overline{K}$ is a Lie subalgebroid of $\Omega \to G$ (and an ideal in the sense of [HM1, §4]). Form the pullback vector bundle (not the pullback Lie algebroid) of $\overline{K}$ over $1: B \to G$; this pullback is $\{\xi \in \Omega \mid \exists b \in B: \widetilde{\alpha}(\xi) = 0_b,\ \widetilde{q}(\xi) = 1_b\}$ and identifies naturally with $K$. We can now put a Lie algebroid structure on $K$ by mimicking the construction of the Lie algebroid of a differentiable groupoid ([M1, III§3]).

Given $X \in \Gamma K$, define $\overline{X} \in \Gamma \overline{K}$ by $\overline{X}(g) = X(\beta g)\widetilde{0}_g$. Define $\xi \in \Gamma\overline{K}$ to be *$G$-equivariant* if $\xi(gh) = \xi(g)\widetilde{0}_h$ for all $g, h \in G$ with $\alpha(g) = \beta(h)$. Clearly $\overline{X}$, for $X \in \Gamma K$, is $G$-equivariant. Observe that for any $\xi \in \Gamma\overline{K}$ there is a well-defined section $\xi * \widetilde{0}: (h, g) \mapsto (\xi(h), \widetilde{0}_g)$ of the pullback Lie algebroid $\Omega * \Omega \to G * G$ (where $\Omega * \Omega$ is the pullback of $\widetilde{\alpha}$ and $\widetilde{\beta}$, and $G * G$ is the pullback of $\alpha$ and $\beta$), and that $\xi$ is $G$-equivariant if and only if $\xi * \widetilde{0} \overset{\kappa}{\sim} \xi$. It follows from this that the bracket of $G$-equivariant sections of $\overline{K}$ is $G$-equivariant, and so we can define

$$[\overline{X}, \overline{Y}] = \overline{[X, Y]} \tag{22}$$

for $X$, $Y \in \Gamma K$. For the anchor, first observe that if $\xi \in \Gamma\overline{K}$ is $G$-equivariant, then $\widetilde{a}(\xi) \in \Gamma T^{\alpha}G$ is right-invariant and is therefore equal to $\vec{Z}$ (in the notation of [M1, III §3]) for a unique $Z \in \Gamma AG$; denote this $Z$ by $\partial_{AG}(X)$, so that we have defined $\partial_{AG}: K \to AG$ by $\overrightarrow{\partial_{AG}(X)} = \widetilde{a}(\overline{X})$. Now let the anchor for $K$ be the composite $a_K = a_G \circ \partial_{AG}: K \to TB$. It is easy to check that $K$, with this anchor, and the bracket structure (22), is a Lie algebroid on $B$, and $\partial_{AG}: K \to AG$ is a morphism of Lie algebroids over $B$. Further, the restriction of $\widetilde{\beta}: \overline{K} \to A$ to $K \to A$ is a morphism of Lie algebroids over $B$, and we denote it by $\partial_A$.

**Definition 5.1.** *With the above notation, $K$ is the* core Lie algebroid *of* $(\Omega; G, A; B)$, *and $K$ together with $\partial_A: K \to A$ and $\partial_{AG}: K \to AG$ is the* core diagram *of* $(\Omega; G, A; B)$. ∎

**Proposition 5.2.** *With the above notation, $\partial_A: K \to A$ is a surjective submersion if and only if $(\widetilde{\chi}: \Omega \to A \times A, \chi_G \to B \times B)$ is a fibration of Lie algebroids. Further, $\partial_{AG}: K \to AG$ is a surjective submersion if $(\widetilde{a}: \Omega \to TG,\ a: G \to TB)$ is a fibration of Lie groupoids.*

PROOF: If $(\widetilde{\chi}, \chi)$ is a fibration, then $\widetilde{\chi}^*: \Omega \to \chi^*(A \times A)$ is a surjective submersion. Within $\chi^*(A \times A) = \{(g, Y, X) \mid \chi(g) = q_{A \times A}(Y, X)\}$ consider the embedded submanifold $E = \{(1_b, Y, 0_b) \mid X \in A, b = q(X)\}$. Now $K$ is the complete inverse image of $E$ under $\chi^*$ and so it follows that the restriction of $\chi^*$ to $K \to E$, which may be identified with $\partial_A: K \to A$, is a surjective submersion.

Conversely, if $\partial_A$ is a surjective submersion, let $P$ denote the pullback of $\chi: G \to B \times B$ and $q \times q: A \times A \to B \times B$. Then there is a commutative diagram, as in Figure 17, and since

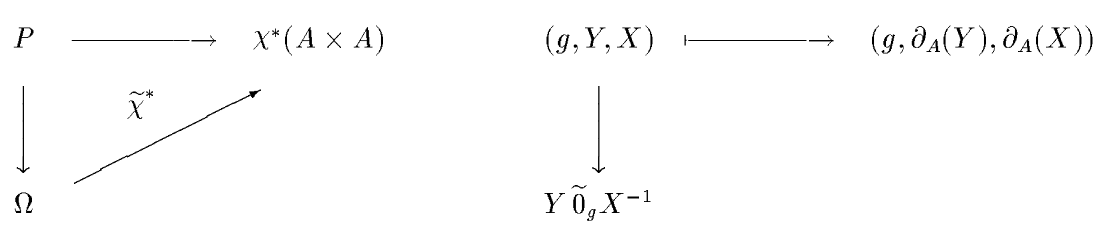



Figure 17.

the horizontal arrow is a surjective submersion, it follows that $\widetilde{\chi}^*$ is also.

Lastly, if $(\widetilde{a}, a)$ is a fibration of Lie groupoids, then $\widetilde{a}^*: \Omega \to a^*(TG)$ is a surjective submersion. Within $a^*(TG) = \{(X, Y) \in A \times TG \mid a(X) = T(\alpha)(Y)\}$ consider $E = \{(0_b, Y) \mid Y \in T(G)_{1_b},\ b \in B\}$. Now $K$ is the complete inverse image of $E$ under $\widetilde{a}^*$, and $\widetilde{a}^*: K \to E$ identifies with $\partial_{AG}$. ∎

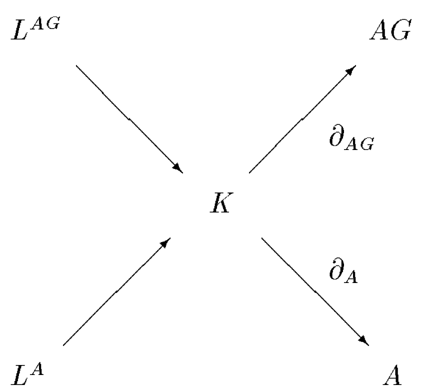
Compare [BrM, 2.4]. It is not clear whether the converse of the last part of 5.2 holds.

**Proposition 5.3.** *Suppose that* $X,\ Y \in \Gamma K$ *have* $\partial_A(X) = 0$ *and* $\partial_{AG}(Y) = 0$. *Then* $[X, Y] = 0$.

PROOF: Since $a_K(X) = a_K(Y) = 0$ the value of $[\overline{X}, \overline{Y}]$ at $g \in G$ depends only on $\overline{X}(g)$ and $\overline{Y}(g)$. We know from §4 that $\overline{Y} * \widetilde{0}$ is a well-defined section of $\Omega * \Omega \to G * G$ and that $\overline{Y} * \widetilde{0} \overset{\widetilde{\kappa}}{\sim} \overline{Y}$. Since $\partial_A(X) = 0$, the section $\widetilde{0} * \overline{X}$ is also defined, but will not usually project under $\widetilde{\kappa}$ to $\overline{X}$. However, at $(1_{\beta g}, g)$ it is true that $\widetilde{\kappa}(\widetilde{0} * \overline{X})(1_{\beta g}, g) = \overline{X}(g)$ and we may loosely write that $\widetilde{0} * \overline{X} \overset{\widetilde{\kappa}}{\sim} \overline{X}$, since $\widetilde{0}$ and $\overline{X}$ can be extended, for values near to $(1_{\beta g}, g)$, in any convenient fashion. Now from $\widetilde{0} * \overline{X} \sim \overline{X}$ and $\overline{Y} * \widetilde{0} \sim \overline{Y}$ we obtain $[\widetilde{0}, \overline{Y}] * [\overline{X}, \widetilde{0}] \sim [\overline{X}, \overline{Y}]$, and since both brackets on the left have the value $\widetilde{0}$ at $(1_{\beta g}, g)$ it follows that $[\overline{X}, \overline{Y}] = \widetilde{0}$ at $g$. ∎

**Definition 5.4.** *An* $\mathcal{LA}$*-groupoid* $(\Omega; G, A; B)$ *is* groupoid-locally-trivial *if* $(\widetilde{\chi}\colon \Omega \to A \times A,\ \chi\colon G \to B \times B)$ *is an s-fibration of Lie algebroids; it is* algebroid-transitive *if* $(\widetilde{a}\colon \Omega \to TG,\ a\colon G \to TB)$ *is an s-fibration of Lie groupoids; it is a* transitive $\mathcal{LA}$-groupoid *if it is both groupoid-locally-trivial and algebroid-transitive.* ∎

It now follows from 5.2 and 5.3 that the core diagram of a transitive $\mathcal{LA}$-groupoid can be represented in the form of Figure 18,




(23)

Figure 18.

where $L^{AG} = ker(\partial_A)$ and $L^A = ker(\partial_{AG})$ are Lie algebra bundles on $B$ whose images in $K$ commute. As with the case of double groupoids [BrM, §2], a representation $\rho_A$ of $A$ on $L^A$ is defined by $\rho_A(X)(V) = [X', V]$ where $\partial_A(X') = X$, and this is a representation by derivations of $L^A$; we have $\rho_A\colon A \to CDO[L^A]$. Similarly there is a representation $\rho_{AG}$ of $AG$ on $L^{AG}$ defined in terms of (23) by $\rho_{AG}(X)(V) = [X', V]$ where $\partial_{AG}(X') = X$.

**Lemma 5.5.** *With the above notation, (i)* $\rho_{AG} = A(\rho_G)\colon AG \to CDO[L^{AG}]$ *where* $\rho_G$ *is the action of* $G$ *on the Lie algebra bundle* $L^{AG}$ *defined by* $\rho_G(g)(V) = \widetilde{0}_g V \widetilde{0}_{g^{-1}}$*; and, (ii)* $\partial_{AG}(\rho_G(g)(V)) = Ad(g)\partial_{AG}(V)$ *for* $g \in G,\ V \in L^{AG}$ *with* $\alpha(g) = q_K(V)$.

The proof of (i) is similar to that for 5.3, and (ii) is immediate.

**Definition 5.6.** *Let* $G$ *be a locally trivial Lie groupoid on a manifold* $B$ *and let* $A$ *be a transitive Lie algebroid on* $B$. *Then a* transitive $\mathcal{LA}$-core diagram *on* $G$ *and* $A$*, denoted*

$(K; \partial_A, \partial_{AG}, \rho_G)$, *consists of a (necessarily transitive) Lie algebroid* $K$ *on* $B$, *morphisms* $\partial_A: K \to A$ *and* $\partial_{AG}: K \to AG$ *which are surjective submersions, and a representation* $\rho_G$ *of* $G$ *on the Lie algebra bundle* $L^{AG} = ker(\partial_A)$, *such that*

(i) *the kernels* $L^{AG}$ and $L^A = ker(\partial_{AG})$ *commute in* $K$; *that is,* $[X, Y] = 0$ *for* $X \in L^{AG}, Y \in L^A$;

(ii) *the representation* $\rho_{AG}$ *of* $AG$ *on* $L^{AG}$ *given by* $\rho_{AG}(X)(V) = [X', V]$, *where* $\partial_{AG}(X') = X$, *is equal to* $A(\rho_G)$, *the Lie algebroid representation induced from* $\rho_G$;

(iii) $\partial_{AG}(\rho_G(g)(V)) = Ad(g)\partial_{AG}(V)$ *for* $g \in G$, $V \in L^{AG}$ *with* $\alpha(g) = q_K(V)$.

*If* $(K'; \partial_{A'}, \partial_{AG'}, \rho_{G'})$ *is a second transitive* $\mathcal{LA}$*-core diagram, on base* $B'$, *then a* morphism of transitive $\mathcal{LA}$-core diagrams $(K'; \partial_{A'}, \partial_{AG'}, \rho_{G'}) \to (K; \partial_A, \partial_{AG}, \rho_G)$ *consists of morphisms of Lie algebroids* $\varphi_K: K' \to K$, $\varphi_A: A' \to A$ *and a morphism of Lie groupoids* $\varphi_G: G' \to G$, *all over* $\varphi_B: B' \to B$, *such that* $\partial_A \circ \varphi_K = \varphi_A \circ \partial_{A'}$, $\partial_{AG} \circ \varphi_K = A(\varphi_G) \circ \partial_{AG'}$ *and the restriction* $\varphi_K: L^{AG'} \to L^{AG}$ *is equivariant with respect to the actions* $\rho_{G'}$ *and* $\rho_G$. *If* $B' = B, G' = G$ *and* $A' = A$, *and if* $\varphi_B, \varphi_G$ *and* $\varphi_A$ *are all identities, then* $\varphi_K$ *is a* morphism of transitive $\mathcal{LA}$-core diagrams *over* $G$ *and* $A$. ∎

Given a transitive $\mathcal{LA}$-core diagram as in 5.6, we construct a transitive $\mathcal{LA}$-groupoid having this core diagram. Let $\Theta$ denote the $\mathcal{LA}$-groupoid $q_K^{**}(G) \approx (K \times K) \ltimes \chi$ constructed from $\partial_{AG}: K \to AG$ as in 4.7. Then the adjoint bundle of the Lie algebroid structure on $\Theta$ is

$$L\Theta = \{(Y, g, X) \in \Theta \mid \overrightarrow{\partial_{AG}(Y)}(g) = -T(\sigma)(\overrightarrow{\partial_{AG}(X)}(g^{-1}))\}.$$

Define $N' = \{(W, g, V) \in \Theta \mid W, V \in L^{AG}\}$. For $V \in L^{AG}$ one has $\partial_{AG}(V) \in LG$, the adjoint bundle of $G$, and so the formula $T(\sigma)_{g^{-1}} = -T(L_g) \circ T(R_g)$, valid on Lie groups, may be applied. It follows that $N' = \{(W, g, V) \in K \times G \times K \mid W, V \in L^{AG}, qW = \beta g, \alpha g = qV,\ \partial_{AG}(W) = Adg\partial_{AG}(V)\}$. Now define $N = \{(W, g, V) \in N' \mid W = \rho_G(g)(V)\}$. A calculation shows that $N'$, and thence $N$, are ideals of the Lie algebroid structure on $\Theta$ (in the sense of [M1, IV 1.9]). Let $Q$ be the quotient Lie algebroid $\Theta/N$ on $G$, and denote the coset of $(Y, g, X)$ modulo $N$ by $< Y, g, X >$. Then $Q$ admits a groupoid structure with base $A$ defined by

$$\widetilde{\beta}(< Y, g, X >) = \partial_A(Y), \qquad \widetilde{\alpha}(< Y, g, X >) = \partial_A(X),$$

$$< Y_2, g_2, X_2 > < Y_1, g_1, X_1 > = < Y_2, g_2 g_1, X_1 + \rho_G(g^{-1})(X_2 - Y_1) > .$$

Proceeding as in [BrM, §2], one can now show that $(Q; G, A; B)$ is a transitive $\mathcal{LA}$-groupoid whose $\mathcal{LA}$-core diagram is precisely the given $(K; \partial_A, \partial_{AG}, \rho_G)$. On the other hand, if $(K; \partial_A, \partial_{AG}, \rho_G)$ is the $\mathcal{LA}$-core diagram of a transitive $\mathcal{LA}$-groupoid $(\Omega; G, A; B)$, then the map $\Theta \to \Omega$, $(Y, g, X) \mapsto Y\widetilde{0}_g X^{-1}$ induces an isomorphism of $\mathcal{LA}$-groupoids $Q \to \Omega$ which preserves $G$ and $A$. To summarize, we have the following theorem.

**Theorem 5.7.**

(i) *The core diagram of a transitive* $\mathcal{LA}$*-groupoid is a transitive* $\mathcal{LA}$*-core diagram.*

(ii) *Given a locally trivial Lie groupoid $G$ and a transitive Lie algebroid $A$ on a common base $B$, and given a transitive $\mathcal{LA}$-core diagram $(K; \partial_A, \partial_{AG}, \rho_G)$ for $A$ and $G$, there is a transitive $\mathcal{LA}$-groupoid, unique up to isomorphisms which preserve $A$ and $G$, whose $\mathcal{LA}$-core diagram is $(K; \partial_A, \partial_{AG}, \rho_G)$.*

(iii) *These two constructions are mutually inverse equivalences between the categories of transitive $\mathcal{LA}$-core diagrams and transitive $\mathcal{LA}$-groupoids.*

■

**Examples 5.8.** Given a locally trivial double Lie groupoid $(S; H, V; B)$ with core diagram $(K, \partial_H, \partial_V)$ and associated actions $\rho_H$, $\rho_V$ as in [BrM], the $\mathcal{LA}$-core diagram of the $\mathcal{LA}$-groupoid $(A_V S; H, AV; B)$ is $(AK, A(\partial_H), A(\partial_V), \rho_H)$, obtained by applying the Lie functor to the core diagram of $S$ in the obvious way.

Given a locally trivial Lie groupoid $G \rightrightarrows B$, the core Lie algebroid of $(TG; G, TB; B)$ is $AG$ itself, with $\partial_{TB}: AG \to TB$ the anchor and $\partial_{AG}$ the identity; the action of $G$ is the standard adjoint action $Ad$ of $G$ on its adjoint bundle $LG$. Given a transitive Lie algebroid $A$ on $B$, the core Lie algebroid of $(A \times A; B \times B, A; B)$ is $A$, with the identity and anchor maps again; the action in this case is the trivial action of $B \times B$ on the zero bundle over $B$.

Given a locally trivial Lie groupoid $G \rightrightarrows B$ and a transitive Lie algebroid $A$ on $B$, the $\mathcal{LA}$-groupoid $\chi^{**}(A \times A) \approx a^{**}TG$ of 4.5 has core Lie algebroid $AG \underset{TB}{\oplus} A$, the direct sum in the category of transitive Lie algebroids over $B$ ([M1, III 2.18]), with $\partial_{AG}$ and $\partial_A$ the canonical projections. The action of $G$ on $LG = ker(\partial_{AG})$ is the standard adjoint action.

Given a locally trivial Lie groupoid $G \rightrightarrows B$ and a transitive Lie algebroid $A$ on $B$, and given a surjective submersion $\varphi: A \to AG$, let $\Omega$ be the $\mathcal{LA}$-groupoid $q^{**}(G) \approx (A \times A) \ltimes \chi$ of 4.7. The core Lie algebroid of $\Omega$ is $A$ itself, with $\partial_A$ the identity and $\partial_{AG} = \varphi$. The action of $G$ on $L^{AG} = B \times 0$ is the trivial action. ■

The classification of transitive $\mathcal{LA}$-groupoids by their core diagrams permits a simple description of connections in the Lie algebroid structure which satisfy a minimal compatibility condition with the groupoid structure.

**Definition 5.9.** *Let $(\Omega; G, A; B)$ be a transitive $\mathcal{LA}$-groupoid. An* algebroid-connection *in $\Omega$ is a pair $(\widetilde{\gamma}, \gamma)$ of vector bundle morphisms, $\widetilde{\gamma}: TG \to \Omega$ over $G$ and $\gamma: TB \to A$ over $B$, with $\widetilde{\chi} \circ \widetilde{\gamma} = id$ and $\chi \circ \gamma = id$, such that $(\widetilde{\gamma}; id, \gamma; id): (TG; G, TB; B) \to (\Omega; G, A; B)$ is a morphism of $\mathcal{VB}$-groupoids. We also say that such a $\widetilde{\gamma}$ is an* algebroid-connection *over $\gamma$.* ■

Thus $\widetilde{\gamma}$ and $\gamma$ are to be connections in the Lie algebroids $\Omega$ on $G$ and $A$ on $B$ which together form a morphism of the groupoid structures. Since $(\widetilde{\gamma}; id, \gamma; id)$ is a morphism of $\mathcal{VB}$-groupoids, it induces a morphism of core vector bundles $\gamma_K: AG \to K$ with $\partial_A \circ \gamma_K = \gamma \circ a_G$, and from $\widetilde{\chi} \circ \widetilde{\gamma} = id$ it follows that $\partial_{AG} \circ \gamma_K = id$. Thus $\gamma_K: AG \to K$, $\gamma: TB \to A$, $id: AG \to AG$ satisfy all the conditions of a morphism of $\mathcal{LA}$-core diagrams except bracket preservation. We call $\gamma_K$ the *core connection* corresponding to $(\widetilde{\gamma}, \gamma)$.

**Theorem 5.10.** *Let $(\Omega; G, A; B)$ be a transitive $\mathcal{LA}$-groupoid with core diagram $(K; \partial_A, \partial_{AG}, \rho_G)$, and let $\gamma: TB \to A$ be a connection in $A$.*

(i) *The assignation described above of a core connection to an algebroid-connection is a bijective correspondence between algebroid-connections $\widetilde{\gamma}: TG \to \Omega$ over $\gamma$ and vector bundle morphisms $\gamma_K: AG \to K$ such that $\partial_A \circ \gamma_K = \gamma \circ a_G$ and $\partial_{AG} \circ \gamma_K = id$.*

(ii) *Further, if $\gamma$ is flat then $\widetilde{\gamma}$ is flat if and only if the corresponding core connection $\gamma_K$ is a morphism of Lie algebroids.*

PROOF: (i) This is most easily treated by observing that 5.7 remains true if the bracket structures are systematically ignored so that all Lie algebroid structures are replaced by "vector bundles with anchor". A $\gamma_K$ satisfying the conditions of the theorem then defines a morphism of core diagrams for such structures, and the corresponding map $TG \to \Omega$ is the desired $\widetilde{\gamma}$. Since $\widetilde{\chi}: \Omega \to TG$ induces $\partial_{AG}: K \to AG$, it follows from $\partial_{AG} \circ \gamma_K = id$ that $\widetilde{\chi} \circ \widetilde{\gamma} = id$.

Alternatively, note that every $Z \in T(G)_g$, $g \in G$, can be written as $\vec{Y}(g) \bullet X^{-1}$ for suitable $Y \in AG|_{\beta g}$, $X \in AG|_{\alpha g}$ and define $\widetilde{\gamma}(Z) = \gamma_K(Y)\widetilde{0}_g\, \gamma_K(X)^{-1}$. It can be checked directly that this is well-defined and has the required properties.

(ii) This follows directly from 5.7 since an algebroid-connection with both $\widetilde{\gamma}$ and $\gamma$ flat is a morphism of $\mathcal{LA}$-groupoids. ■

In particular, algebroid-connections exist. In the case of the $\mathcal{LA}$-groupoids of a double Lie groupoid $(S; H, V; B)$, core connections $AH \to AK$ can be identified with suitably equivariant connections in the principal bundles $K_b(H_b, M_b^V, \partial_H)$, for $b \in B$, in the manner of [M2, 3.2]. The connection theory of double Lie groupoids will be developed further elsewhere.

Finally we complete a circle of ideas begun in [HM1, §4] and [HM2]. In [HM2] it was shown that $s$-fibrations of Lie groupoids are completely characterized by an elaboration of the standard concept of kernel, called kernel systems: the proof proceeded by showing that both $s$-fibrations and kernel systems are equivalent to what were called differentiable congruences; differentiable congruences are precisely double Lie subgroupoids of double groupoids of the form $(G \times G; B \times B, G; B)$. In [HM1, §4] a similar characterization of $s$-fibrations of Lie algebroids in terms of (what are called below) ideal systems was given. We now show that the corresponding concept of Lie algebroid congruence is equivalent to that of ideal system.

**Definition 5.11.** *Let $A$ be a Lie algebroid on base $B$. An* ideal system *for $A$ is a triple $\mathcal{K} = (K, R(f), \theta)$ where $K$ is a vector subbundle of $A \to B$, $R(f)$ is the kernel pair of a surjective submersion $f: B \to B'$, and $\theta$ is a linear action of the Lie groupoid $R(f) \rightrightarrows B$ on $A/K \to B$ such that*

(i) *if $X, Y \in \Gamma A$ are $\theta$-stable, then $[X, Y]$ is $\theta$-stable;*

(ii) *if $X \in \Gamma K$, $Y \in \Gamma A$ and $Y$ is $\theta$-stable, then $[X, Y] \in \Gamma K$;*

(iii) *the anchor $a: A \to TB$ maps $K$ into $T^f B$, the vertical subbundle tangent to the fibres of $f$;*

(iv) *the map $A/K \to TB/T^f B$ induced by $a$ is $R(f)$-equivariant with respect to $\theta$ and the canonical action of $R(f)$ on $TB/T^f B \cong f^*TB'$.*

■

Here a section $X \in \Gamma A$ is called $\theta$-stable if its class $\overline{X} \in \Gamma(A/K)$ is stable under the action $\theta$. Definition 5.11 is from [HM1, 4.4], where ideal systems were simply called *ideals* of $A$.

**Definition 5.12.** *Let $A$ be a Lie algebroid on $B$. A* Lie algebroid congruence *for $A$ is a pair $(\Omega, R)$ where $\Omega \subseteq A \times A$ and $R \subseteq B \times B$ are such that*

**(i)** *$\Omega$ is a closed embedded wide Lie subgroupoid of the pair groupoid $A \times A$ on $A$, and $R$ is a closed embedded wide Lie subgroupoid of the pair groupoid $B \times B$ on $B$;*

**(ii)** *$\Omega$ is a Lie subalgebroid of the Cartesian square Lie algebroid $A \times A$ on $B \times B$;*

**(iii)** *the map $\Omega \to R * A$, $(Y, X) \mapsto (qY, qX, X)$ is a surjective submersion to the pullback of $q\colon A \to B$ and the projection $R \to B$, $(b, a) \mapsto a$.*

This definition is modelled on [HM2, 2.2]. Clearly a Lie algebroid congruence is precisely a sub$\mathcal{LA}$-groupoid $(\Omega; R, A; B)$ of the $\mathcal{LA}$-groupoid $(A \times A; B \times B, A; B)$ of 4.4. Further, closed embedded wide Lie subgroupoids of pair groupoids are known, by the Godement criterion, to be precisely the kernel pairs of surjective submersions.

**Theorem 5.13.** *Let $A$ be a Lie algebroid on $B$.*

**(i)** *Let $\varphi\colon A \to A'$, $f\colon B \to B'$ be an $s$-fibration of Lie algebroids. Then the kernel pairs of $\varphi$ and $f$ form a Lie algebroid congruence $(R(\varphi), R(f))$ for $A$.*

**(ii)** *Let $(\Omega, R)$ be a Lie algebroid congruence for $A$. Denote the core Lie algebroid of $(\Omega; R, A; B)$ by $K$, and write $R = R(f)$ for a surjective submersion $f\colon B \to B'$. Given $(b, a) \in R$ and $X \in A_a$, define $\theta(b, a)(\overline{X}) = \overline{Y}$, where $Y$ is any element of $A_b$ with $(Y, X) \in \Omega$. Then $\theta$ is a well-defined action of $R \rightrightarrows B$ on the quotient vector bundle $A/K \to B$ and $(K, R(f), \theta)$ is an ideal system for $A$.*

**(iii)** *Together with the bijective correspondences of [HM1, 4.5]. the correspondences of (i) and (ii) give a commuting triangle of equivalences between the concepts of $s$-fibration of Lie algebroids, ideal systems for Lie algebroids, and Lie algebroid congruences.*

PROOF: (i) From the fact that $\varphi \times \varphi\colon A \times A \to A' \times A'$, $f \times f\colon B \times B \to B' \times B'$, $\varphi\colon A \to A'$ and $f\colon B \to B'$ form a morphism of $\mathcal{LA}$-groupoids, it follows that $R(\varphi)$ and $R(f)$ satisfy the first two conditions of 5.11. Applying [HM2, 1.1] to

$$
\begin{array}{ccccc}
R(\varphi) & \xrightarrow{\;pr\;} & A & \xrightarrow{\;\varphi\;} & A' \\
{\scriptstyle q \times q}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\
R(f) & \xrightarrow[\;pr\;]{} & B & \xrightarrow[\;f\;]{} & B'
\end{array}
$$

it follows, from the fact that $\varphi^*\colon A \to f^*A'$ is a surjective submersion, that $R(\varphi) \to R(f) * A$ is also.

(ii) The core Lie algebroid $K$ may be identified with $\{Y \in A \mid (Y, 0_{qY}) \in \Omega\}$; note that the maps $\partial_A$ and $\partial_{A(R(f))}$ are the inclusion into $A$ and the anchor $K \to T^f B$. Given

$(b,a) \in R(f)$ and $X \in A_a$, the existence of $Y \in A_b$ with $(Y,X) \in \Omega$ follows from condition (iii) of 5.12. If $Y' \in A_b$ also has $(Y',X) \in \Omega$, then $(Y'-Y,0_a) = (Y',X)-(Y,X) \in \Omega$ so $(Y'-Y,0_b) = (Y'-Y,0_a)(0_a,0_b) = (Y'-Y,0_a)\widetilde{0}_{(a,b)} \in \Omega$ and therefore $Y'-Y \in K$. It follows that $\theta(b,a)(\overline{X}) = \overline{Y}$ is a well-defined linear action of $R(f)$ on $A/K \to B$, smoothness following as in [HM2, 3.2(ii)]. Further, it is now clear that a section $X \in \Gamma A$ is $\theta$-stable if and only if $(X(b),X(a)) \in \Omega$ for all $(b,a) \in R$.

Suppose then, that $X,Y \in \Gamma A$ are $\theta$-stable. Let $(X,X)$ and $(Y,Y)$ denote the corresponding sections of $\Omega \to R$. Then, because $\Omega \to R$ is a Lie subalgebroid of $A \times A \to B \times B$, it follows that $([X,Y],[X,Y]) = [(X,X),(Y,Y)]$ is a section of $\Omega \to R$. So $[X,Y]$ is $\theta$-stable.

Next observe that $X \in \Gamma A$ belongs to $\Gamma K$ if and only if $(X(b),0_a) \in \Omega$ for all $(b,a) \in R$. So, for $X \in \Gamma K$ there is a well-defined section $(X,0)$ of $\Omega \to R$. If now $X \in \Gamma K$ and $Y \in \Gamma A$ is $\theta$-stable, then $([X,Y],0) = [(X,0),(Y,Y)] \in \Gamma_R\Omega$ and it follows that $[X,Y] \in \Gamma K$.

As already noted, the anchor $K \to TB$ takes values in $T^fB$. To verify condition (iv) of 5.11, observe that the anchors give a morphism of $\mathcal{LA}$-groupoids $\widetilde{a}\colon \Omega \to T(R(f)) = R(T(f))$, $a\colon A \to TB$, $id\colon R(f) \to R(f)$, $id\colon B \to B$, and that if the construction of $\theta$ above is applied in the same way to the $\mathcal{LA}$-groupoid $(R(T(f)); R(f), TB; B)$, one obtains the canonical action of $R(f)$ on $TB/T^fB$. The equivariance of $A/K \to TB/T^fB$ then follows.

■

Thus the core of a congruence is the kernel.

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