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Prologue

Groupoids possess many of the features which give groups their power and importance, but apply in situations which lack the symmetry which is characteristic of group theory and its applications. Though only developed since the mid 20th century, the modern concept of Lie groupoid is as much entitled as is the familiar concept of Lie group to be regarded as the rigorous formulation of the 19th century notion which went under the then vague term 'continuous group of local transformations'; a case could be made that the modern concept of Lie group has been a transitional stage in the evolution of the notion of Lie groupoid.

Groups arise primarily, though not exclusively, in connection with symmetry; that is, as sets of automorphisms of geometric or other mathematical structures. From this viewpoint, groupoids are the natural formulation of a symmetry system for objects which have a bundle structure. The most immediate illustration from geometry is to think of a tangent bundle: with each tangent space there is at first associated a general linear group, and the presence of a geometric structure on the manifold — such as a metric, or a complex structure — is reflected in the replacement of this group by a subgroup — such as the orthogonal or complex linear group. But a tangent space is a linear approximation to the manifold only near a single point and any geometrical study will involve moving from point to point within the manifold. This being so, it is necessary to consider the isomorphisms between different tangent spaces, and one is thus led to the groupoid of all linear isomorphisms between all tangent spaces, or to a subgroupoid of isomorphisms which preserve a given additional structure.

A distinct but equally important source of groupoid theory is the fundamental groupoid of a space. Here one has a similar relationship between fundamental group and fundamental groupoid: the fundamental group at a specific point consists of (homotopy classes of) loops at the chosen point, whereas in the fundamental groupoid one considers (homotopy classes of) paths between arbitrary points. The fundamental groupoid acts upon the system of fundamental groups (and on the systems of higher homotopy groups).

Both these cases are, in groupoid terms, somewhat special. They are locally trivial in a sense which is similar to that for fibre bundles, and because of this there is a compromise position, which yields a principal bundle. In the first case, rather than considering the system of all isomorphisms between the tangent spaces of a manifold M, one fixes a reference point m_0 and considers isomorphisms from $T_{m_0}(M)$ to all other tangent spaces; these can be readily identified with isomorphisms from a standard reference space $\mathbb{R}^n$, and one arrives at the full frame bundle or a reduction of it. In the second case, fixing a reference point and considering (homotopy classes of) paths from it to arbitrary points within the manifold yields the universal cover, with its structure as a principal bundle over the given space. For a general Lie groupoid, however, not locally trivial, choosing a point of the base manifold and considering the arrows which radiate from it, yields a structure which is not equivalent to the original groupoid.

It was at one time very common to find groupoid theory criticised as an artificial generalization of group theory. Personally, I find generalization an always useful technique for understanding basic concepts; certainly as useful as the dual study of examples. A well–chosen generalization supplies for a theory the effect of a frame on a painting, and enables it to be looked at 'from the outside' as it were. Nonetheless, the case for Lie groupoids and Lie algebroids does not rely on such considerations. I outline here four of the most important.

(I) The Lie theory of Lie groups and Lie algebras is one of the cornerstones of 20th century mathematics, and it is hard to disentangle the importance of Lie groups and Lie algebras as encoders of symmetry from the importance of the classification results for Lie algebras. There is no comparable classification of Lie algebroids — and I personally do not believe that such a classification is possible — but the basic processes of Lie theory carry over to Lie groupoids and Lie algebroids and provide a unified approach to many of the fundamental constructions in first–order differential geometry. Differential geometry is, after all, the study of geometry by means of differentiation — or linearization — and the process of taking the Lie algebroid of a Lie groupoid demonstrates that this basic construction has no necessary relationship with the case of groups and all the familiar symmetry which is present in that case.

The three classical results of Lie theory — the integrability of morphisms, the integrability of subobjects, and the integrability of abstract Lie algebras — all generalize to meaningful questions in the context of Lie groupoids and Lie algebroids. The answers are of course more complex, but what is important is that they embody results of intrinsic geometric interest, both known and new.

In Part II I give a detailed exposition of this aspect in the case of locally trivial Lie groupoids and transitive Lie algebroids. In this case the integrability of morphisms embodies the triviality of a bundle which has a flat connection and a simply–connected base; the integrability of Lie subalgebroids embodies the Ambrose–Singer and Reduction Theorems; and the integrability question for abstract transitive Lie algebroids gives criteria for the existence of connections when curvature data is prescribed. Expressed in terms of principal bundles these results fit into no clear framework.

(II) The relationship between the results of connection theory mentioned in **(I)** and the classical Lie theory of Lie groups and Lie algebras is concealed — very effectively — by the awkward nature of the algebra of principal bundles. Indeed, there hardly exists a recognizable algebraic theory of principal bundles, perhaps in part because there has been no clear model on which to build it: principal bundles do not behave very much like modules, nor like vector bundles. Nonetheless, it is possible to develop an algebraic theory of groupoids (set–theoretic and Lie) to an extent which may be surprising at first. This algebraic theory begins by being modelled on standard group theory, but diverges from it in some important respects; it is possible, for example, to characterize many algebraic constructions by classes of morphisms in a way which is impossible for ordinary groups. These constructions have analogues for Lie algebroids and together they form a technique of great value in treating geometric questions involving linearization and globalization. This process is described in detail in Chapters 2 and 4.

All frame groupoids and all fundamental groupoids are locally trivial. Lie groupoids which are locally trivial are symmetry structures of vector bundles — or of more general fibre bundles — and it is precisely the class of locally trivial groupoids for which the concept of principal bundle provides an alternative. It is intrinsic to the nature of general Lie groupoids that they are not determined by their vertex structure in the way that is true in the locally trivial case.

It is sometimes suggested that the notion of symmetry should be extended so that general Lie groupoids may be regarded as symmetry structures. This seems to me debatable; the concept of symmetry as it passes from ordinary usage into mathematics retains features which link it firmly to group symmetry. To extend the word to encompass arbitrary groupoids — in effect, to treat every equivalence relation as if it were the orbit relation of a group action — robs the word of much of its significance. However, one could distinguish groupoids from groups by the possibility of a continuous variation of symmetry.[1]

(III) Large families of Lie groupoids which are not locally trivial arise naturally: as the symplectic realizations of Poisson manifolds, in the study of non–transitive Lie group actions, and in foliation theory, for example. Whereas many basic constructions have been known for general Lie groupoids for a number of years, it is only very recently that strong results have been obtained in the general case. The fact that practical results and working techniques now exist for general Lie groupoids is a compelling argument for abandoning the concentration on the principal bundle approach in the locally trivial case.

Most of Part I, and most of Part III, applies to general Lie groupoids and Lie algebroids. However, it has not been possible to give a complete account of all recent developments, and some topics are only described in outline in the Appendix.

(IV) Where future developments are concerned, the most important distinction between groupoids and groups lies in the existence of higher–dimensional forms of the concept. It is widely appreciated that iterating the group concept by considering a group object in the category of groups leads to nothing new: a group object in the category of groups is merely an abelian group, a result which is the abstract form of the fact that the higher–order homotopy groups are all abelian. However, a groupoid object in the category of groupoids is a genuinely new and different object. These double groupoids arise naturally in Poisson geometry and may be regarded as a semi–classical form of the use of multiple category theory in quantization.

This theory lies largely beyond the present book, though some aspects of double structures are treated in Part III. Further references are given in the Appendix.

[1] Plotinus (A.D. 205–269/70) defined beauty as symmetry irradiated by life: 'There must be symmetry, achieved by the perfect realisation of geometrical possibilities. There must be a feeling of movement, for movement meant life.' [Runciman, 2004].

Introduction

As with many books, this one is best read piecewise backwards. In describing the contents, I accordingly begin with Part III.

Groupoid theory was transformed in the mid 1980s by the introduction of the notion of symplectic groupoid and the methods of Poisson geometry. The announcement by Weinstein [1987] and the seminar notes of Coste, Dazord, and Weinstein [1987] on symplectic groupoids and Poisson geometry became available about late 1986. In fact a similar approach to the use of groupoid structures in Poisson geometry had been given by Karasëv [1989] in papers deposited in VINITI in Moscow in 1981 but not generally available until much later. The two papers of Zakrzewski [1990a,b], gave a third and independent treatment. In most of the discussion in this Introduction I will treat these three very different approaches as if they were a single body of work.

The work of these authors transformed both the subject and the applications of Lie groupoid and Lie algebroid theory. Until that time only the case of locally trivial Lie groupoids and transitive Lie algebroids was well–understood. Despite the very general programme and results announced by Pradines in four short notes [1966], [1967a], [1967b], [1968], and some isolated work on specific aspects of general Lie groupoids and Lie algebroids, there seemed to be little compelling reason to understand the very difficult general theory.

The reciprocal influence — the importance of groupoid theory in Poisson geometry — is based on two fundamental observations. Firstly, that the Poisson bracket of 1–forms on a Poisson manifold P makes the cotangent bundle T^*P a Lie algebroid — this fact in itself was found by a number of authors; the survey of Huebschmann [1990] gives a detailed account of the history. Secondly, on the relationship between realizations of Poisson manifolds and groupoid structures. A realization[1] of a Poisson manifold P is a surjective submersion $S \to P$ which is a Poisson map from a symplectic manifold S to P. The simplest interesting example is that the linear Poisson structure on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ has a realization $\mathscr{R}\colon T^*G \to \mathfrak{g}^*$ where G is any Lie group integrating $\mathfrak{g}$ and $\mathscr{R}$ is right–translation. That this map, and the corresponding left–translation $\mathscr{L}$, give symplectic realizations had been known for a considerable time: the new and crucial observation in the 1980s was that $\mathscr{R}$ and $\mathscr{L}$ are the source and target maps for a natural groupoid structure on T^*G defined by the coadjoint action, and that the canonical symplectic structure on T^*G is compatible with this groupoid structure in a natural way.

The concept of symplectic groupoid extends this example and links the two fundamental observations tightly together. For any symplectic groupoid Σ with base a Poisson manifold P, the target map is a symplectic realization of P and the source map is a symplectic realization of the opposite structure. Thus Σ with its symplectic structure may be regarded as a desingularization of P with its Poisson structure. Most remarkably, the Lie algebroid $A\Sigma$ of the Lie groupoid structure and the cotangent Lie algebroid T^*P of the Poisson manifold P are canonically isomorphic. Thus the realization problem for Poisson manifolds has been reduced to an aspect of a generalized Lie theory. (It is, furthermore, true that for P a Poisson manifold, any Lie groupoid which integrates T^*P and which is suitably connected, has a canonical symplectic structure making it a symplectic groupoid with base P.) In particular, integrating a Lie algebra $\mathfrak{g}$ and finding a symplectic realization of $\mathfrak{g}^*$ are equivalent problems.

Thus the Lie theory of Lie groupoids and Lie algebroids embodies the classical Lie theory of Lie groups and Lie algebras not only in its standard form, but also in the dual form which is a special case of the relationship between symplectic groupoids and Poisson manifolds.

Symplectic groupoids are only rarely locally trivial, and their behaviour is very far removed from the features of the locally trivial case. The existence of a symplectic groupoid structure is a very strong constraint on a groupoid. While symplectic groupoids provided a definite reason for studying Lie theory without the restriction of local triviality, it has until recently proved difficult to construct large families of examples of symplectic groupoids.

[1] This is actually a *full* realization in the terminology of Weinstein [1982]; I will not consider realizations in which the map is not a surjective submersion.

The importance of the work of Karasev, Weinstein and Zakrzewski for Lie groupoid and Lie algebroid theory themselves rests primarily on two further developments.

Firstly, one consequence is a duality between Lie algebroids and vector bundles with a Poisson structure which respects the linear and bundle structures: this book calls these simply *Poisson vector bundles*. This duality, given in detail by Courant [1990], extends the classical duality between Lie algebras and linear Poisson structures, and also gives a clear meaning to the statement that the canonical symplectic structure on a cotangent bundle is the dual of the bracket of vector fields.

This duality is nontrivial in the specific sense that it takes place outside each of the categories with which it is concerned: one must consider sections of the Lie algebroid and functions (or 1–forms) on the dual bundle. Taken together with the process which associates the cotangent Lie algebroid to any Poisson manifold, this means that there are two processes which pass between the Poisson category and the Lie algebroid category, and although these are far from giving an equivalence, they frequently allow problems on one side to be usefully transformed into problems in the other. Because these processes are non–trivial and not genuinely inverse, they often deliver a substantial benefit.

Secondly, Weinstein [1988] introduced a concept of Poisson groupoid which unites the two extreme cases of symplectic groupoids and Poisson Lie groups. That such a unification would be possible does not seem obvious even in retrospect: symplectic groupoids were introduced as global realizations of Poisson manifolds, whereas the concept of Poisson Lie group was introduced by Drinfel'd [1983] as a semi–classical form of the notion of quantum group, and was then seen to also provide a valuable tool in work on complete integrability. Nonetheless the concept of Poisson groupoid provides a continuum of structures linking symplectic groupoids to Poisson Lie groups. Poisson groupoids have been shown to provide an appropriate general framework in which to study the classical dynamical Yang–Baxter equation [Etingof and Varchenko, 1998]. Their infinitesimal form, the concept of Lie bialgebroid defined by myself and Ping Xu [1994] is part of a family of algebraic concepts under development in mathematical physics [Xu, 1999]. Poisson groupoids and Lie bialgebroids have turned out to be crucially important in the study of double Lie groupoids and double Lie algebroids [Mackenzie, 1998].

The account of Poisson groupoids which I give here is a theoretical one — there has been no space to deal with examples beyond the most fundamental. The treatment is new and appears in print here for the first time, though I have had the good fortune to be able to set it out in series of lectures at Utrecht and Amsterdam in 2000, and at Queen Mary, London in 2002.

The crux of this approach is the observation that the compatibility condition between a Poisson structure and a Lie groupoid structure on a manifold G is equivalent to a compatibility condition between the Lie groupoid structure on T^*G induced by the groupoid structure on G, and the anchor of the Lie algebroid structure induced on T^*G by the Poisson structure on G. This condition is of categorical or diagrammatic type — it is of the same nature as the compatibility conditions on double structures in the categorical sense: that the structure maps of one structure be morphisms with respect to the other. In fact the groupoid structure maps in T^*G, for G a Poisson groupoid, are Lie algebroid morphisms, so that T^*G is an $\mathscr{LA}$–groupoid in the terminology of [Mackenzie, 1992]. General $\mathscr{LA}$–groupoids occupy an intermediate place between double Lie groupoids and double Lie algebroids. This double Lie theory is not treated in the present book, but the treatment of Poisson groupoids given here provides much of the background necessary for it.

A more immediate consequence of this treatment of Poisson groupoids is that the basics of symplectic groupoid theory may be developed without any use of genuine symplectic geometry. I deduce the basic properties of symplectic groupoids in §11.5 as an immediate corollary of the general Poisson groupoid theory, merely by imposing the nondegeneracy condition of the Poisson anchor. The first accounts of symplectic groupoid theory made extensive use of nontrivial and genuinely symplectic results, but the approach given here avoids this entirely. This is not, of course, because of any lack of appreciation of the power and importance of symplectic geometry, but to demonstrate that those aspects of symplectic groupoid theory which often surprise people new to the subject, are not in fact consequences of symplectic geometry as such. This procedure extends beyond the aspects treated in this book, and may be applied, for example, to deduce results on symplectic groupoid actions from general Lie theory.

One might call the principle underlying this approach a 'cotangent philosophy' — that where general constructions with Poisson structures are concerned, it is conceptually simpler to work with the cotangent Lie algebroid rather than with the actual manifold. Readers not already acquainted with this approach may at first question the word 'simpler' — there is, for instance, a good deal of apparatus in §11.5. However, the calculus of canonical isomorphisms which I develop here behaves in a direct, categorical fashion, and applies with minor changes to much more general situations.

It was not possible, in a book of reasonable length, to provide a treatment of work on double Lie structures while also covering most of the basic theory and constructions of Lie groupoids and Lie algebroids. What I have been able to do — I trust — is to show in how completely natural a way work with Poisson groupoids (and hence Poisson Lie groups and symplectic groupoids) is clarified by the consideration of double structures. One sees this already when considering Poisson structures on a vector bundle: these can be very easily handled through the associated double vector bundles.

Part III therefore begins in **Chapter 9** with an account of double vector bundles. The notion of double vector bundle has been around for some time — the general notion and an extensive account of much general theory was given by Pradines [1974a], and the double tangent bundle of a manifold and the tangent of a general vector bundle were sometimes used in accounts of connection theory in the 1960s and 1970s [Dieudonné, 1972], [Besse, 1978], [Yano and Ishihara, 1973]. Iterated tangent and cotangent bundles have a well–established place in some treatments of classical mechanics [Tulczyjew, 1989]. The chapter starts with an account of the general concept and progresses immediately to recent results on the duals of a double vector bundle, due independently to myself [Mackenzie, 1999] and to Konieczna and Urbański [1999]. Briefly stated, a double vector bundle may be dualized either along its horizontal structure or its vertical structure, and these two duals are themselves dual. Thus successive dualizations of a double vector bundle return to the original structure not at the second dual, but at the third. This phenomenon is likely to be of great significance in further work on multiple structures.

Applying this result to the tangent double vector bundle of a vector bundle, Chapter 9 recovers the canonical isomorphisms between the cotangents of a vector bundle and of its duals, introduced by myself and Ping Xu [1994, 1998] and the canonical pairing between the tangents of a vector bundle and of its dual. These will be used repeatedly in the subsequent chapters.

Chapter 10 gives the crucial relationships between Poisson structures and Lie algebroids. The basic definitions and properties of Poisson structures are included, but a reader entirely new to Poisson geometry will need to supplement this with other sources. I give the construction of the cotangent Lie algebroid of a Poisson manifold and in §10.3 treat the duality between Poisson vector bundles and Lie algebroids. In §10.2 I briefly consider Poisson cohomology as an example of Lie algebroid cohomology, and describe the associated Batalin–Vilkovisky structures. §10.4 gives the correspondences for subobjects and for morphisms between Poisson structures and Lie algebroids; in principle all concepts of Poisson geometry may be translated to Lie algebroids, and reciprocally.

Chapter 11 treats Poisson groupoids, beginning with a brief resumé of Poisson Lie groups in terms of the Lie groupoid and Lie algebroid structures on T^*G. The techniques normally used for a Poisson group G exploit the fact that the tangent group TG is trivializable as a bundle and is thus a semi–direct product, with respect to the adjoint representation, as a group, Thus one commonly defines a Poisson structure and a Lie group structure to be compatible if the map $G \to \mathfrak{g} \wedge \mathfrak{g}$ derived from the Poisson tensor π is an Ad–cocycle. This formulation fits into no pre–existing framework and it can take some time to build up a feel for how to proceed. Alternatively, one may define a Poisson structure and a Lie group structure to be compatible if the multiplication map is Poisson. This appears to be a compatibility condition of a standard type, but the 'backward flipping' or contravariant nature of Poisson structures means that this intuition can mislead: group inversion is not a Poisson map, but anti–Poisson, and right– and left– translations are neither Poisson nor anti–Poisson; the analogy with more familiar compatibility conditions breaks down. I demonstrate in §11.1 that naturality is restored by working with the structures on the cotangent bundle.

For a general Lie groupoid $G \rightrightarrows P$ there is no simple version of the adjoint and coadjoint representations. The original treatment of Weinstein [1988] was in terms of the coisotropic calculus introduced in the same paper. Here I make systematic use of the structures on the cotangent bundle. Both the tangent bundle TG and the cotangent bundle T^*G are 'double' objects in a categorical sense: they possess a vector bundle structure and a Lie groupoid structure, and the structure maps of each structure are morphisms with respect to the other structure. This kind of compatibility condition has been used in category theory since at least the 1960s, when Ehresmann made what at the time must have seemed to be a complete change of direction from differential geometry to multiple category theory. This use of the word 'double' should be carefully distinguished from its use in work on the Drinfel'd double, though in fact the two usages are related in a non–obvious fashion [Mackenzie, 1998].

The double structure — more precisely the $\mathscr{VB}$–groupoid structure as defined by Pradines [1988] — of TG and T^*G supplies a replacement for the actions which characterize the structure of a tangent or cotangent group. In §11.2 I define the abstract notion of $\mathscr{VB}$–groupoid, a groupoid object in the category of vector bundles, and give the duality which exists for such structures. This extends the duality for double vector bundles which was treated in Chapter 9. The brief §11.3 deduces explicitly the properties of the cotangent groupoid and §11.4 then gives the basic properties of Poisson groupoids.

In §11.5 I first deduce the basic properties of symplectic groupoids as a special case of Poisson groupoids. The main part of the section proves that $T^*G \rightrightarrows A^*G$, for G a general Lie groupoid, is a symplectic groupoid with respect to the canonical symplectic structure on T^*G. The proof proceeds by showing that the canonical isomorphisms R, I and Θ associated with iterated tangent and cotangent bundles respect the relevant groupoid structures. This 'calculus of canonical isomorphisms' can be readily extended to more general situations.

Chapter 12 is a short account of the basic theory of Lie bialgebroids. I take as definition the Schouten calculus formula used by myself and Ping Xu in the original paper [Mackenzie and Xu, 1994] and prove its self–duality by a mixture of the proof given in [Mackenzie and Xu, 1994] and the much improved formulation of Kosmann–Schwarzbach [1995]. §12.2 gives the proof that this definition is equivalent to a morphism criterion which is parallel to the definition of Poisson groupoid. The following §12.3 gives an alternative and often preferable construction of the Poisson structure on the Lie algebroid of a Poisson groupoid. Finally in §12.4 I consider Poisson actions and moment maps for Poisson Lie groups; this is an introduction to a large further subject.

Part II

Part II of the book is devoted to the theory of transitive Lie algebroids and locally trivial Lie groupoids. This case is inextricably bound up with connection theory.

Connection theory was a very large and important area of differential geometry for most of the 20th century; there now seems to be some danger of it falling out of common knowledge. From the introduction of Christoffel symbols and the identities of Bianchi around 1870 to the explosive development of gauge theory about a hundred years later, connection theory was understood to be a fundamental concept in geometry and mathematical physics. In the late 19th century a connection was a structure associated with a surface or higher–dimensional Riemannian manifold; in modern terminology this is the Levi–Civita connection of a Riemannian manifold. The concept was then separated from a metric structure by Weyl in an early attempt at a unified field theory, giving rise to what we now think of as an affine connection in a manifold. Some 20 to 30 years later, in the late 1950s, Ehresmann formulated a very general notion of connection in a principal bundle or locally trivial Lie groupoid. It was this notion, in its principal bundle form, which Kobayashi and Nomizu [1963] put at the start of their classic treatment of the foundations of differential geometry. In the 1970s the treatment of [Kobayashi and Nomizu, 1963, Chap 2] was widely regarded as mysterious and difficult; some 20 or 30 years of gauge theory has lessened that attitude somewhat, but it is still I think the case that the treatment of connection theory in principal bundles is regarded as a less easily assimilated piece of mathematics than one expects of such a fundamental concept. Indeed many treatments still focus almost exclusively on Koszul connections in vector bundles, a concept which is much more easily absorbed. However, to dismiss principal bundles as an optional elaboration of vector bundles is a modern version of the old misconception that the only Lie groups which matter are matrix groups.

The formulation of connection theory in terms of Lie groupoids and Lie algebroids makes clear that connection theory follows the pattern of the Lie theory of Lie groups and Lie algebras. Indeed connection theory ***is***, in a sense which is made precise below, the Lie theory of locally trivial Lie groupoids and Lie algebroids. To justify this statement quickly, recall first of all the basic three theorems[1] of Lie:

[Lie–1] Let G and H be Lie groups with G simply–connected and with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras. Then there is a unique morphism of Lie groups $F\colon G \to H$ which induces φ.

[Lie–2] Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then there is a unique connected Lie subgroup H of G for which the Lie algebra is $\mathfrak{h}$.

[1] These are, of course, modern formulations and do not exactly correspond to the three theorems as stated in, for example, expository books of the early 1900s. Perhaps they should be called the *three integrability theorems of Lie.*

[Lie–3] Given a Lie algebra $\mathfrak{g}$, there is a Lie group G with Lie algebra isomorphic to $\mathfrak{g}$.

Throughout the book, I take all Lie groups and Lie algebras to be finite–dimensional and real, except where explicitly indicated otherwise.

Cognate with these theorems are the following important riders:

[Lie–4] Let G be a Lie group. Then the connected component of the identity element, G_0, is a Lie group and the inclusion $G_0 \to G$ induces the identity map of Lie algebras $\mathfrak{g} \to \mathfrak{g}$.

[Lie–5] Let G be a connected Lie group. Then the universal cover $\widetilde{G}$ is a Lie group and the covering projection $\widetilde{G} \to G$ induces the identity map of Lie algebras $\mathfrak{g} \to \mathfrak{g}$.

I show in Part II that each of these results, **[Lie–1]** to **[Lie–5]**, has an extension to locally trivial Lie groupoids and transitive Lie algebroids, which embodies substantial results for principal bundles or their connection theory. The extension of **[Lie–1]** to locally trivial Lie groupoids (6.2.11) has the same form and implies the following well–known result for principal bundles:

[CT–1] Let $P(M, G)$ be a principal bundle with M simply–connected. If P admits a flat connection, then it is trivializable.

Likewise, the extension of **[Lie–2]** to locally trivial Lie groupoids (6.2.1) has the same form and implies the following result, which combines the Ambrose–Singer and Reduction Theorems for principal bundles:

[CT–2] Let $P(M, G)$ be a principal bundle and let γ be a connection in $P(M, G)$. There is a least Lie subalgebroid of the Lie algebroid constructed from $P(M, G)$ which contains the values of γ and the values of its curvature form, and this Lie subalgebroid is the Lie algebroid corresponding to any holonomy bundle of γ.

The extension of **[Lie–3]** to transitive Lie algebroids is a more complex matter, treated in Chapter 8. For a real valued closed 2–form ω on a manifold M, a classical lemma of Weil states that ω is the curvature of a connection in a circle bundle over M if and only if the Čech class corresponding to the de Rham class of ω is integral. The extension of **[Lie–3]** to transitive Lie algebroids gives a comprehensive generalization of this to 2–forms which take values in an arbitrary Lie algebra bundle. See 8.3.9 for a precise statement.

The two riders extend to locally trivial Lie groupoids in a straightforward fashion. The corresponding results for principal bundles are as follows:

[CT–4] Let $P(M,G)$ be a principal bundle and let P_0 be any connected component of P. Then there is an open subgroup G_1 of G such that $P_0(M,G_1)$ is a principal subbundle of $P(M,G)$, and the inclusion $P_0 \to P$ induces an isomorphism of the corresponding Lie algebroids.

[CT–5] Let $P(M,G)$ be a principal bundle with P connected. Then the universal cover $\widetilde{P}$ has a principal bundle structure $\widetilde{P}(M,H)$ for a certain Lie group H, which is an extension of G, such that the covering projection $\widetilde{P} \to P$ is a morphism of principal bundles over M and induces an isomorphism of the corresponding Lie algebroids.

Thus the basic Lie theorems for locally trivial Lie groupoids and transitive Lie algebroids give results for connection theory. The proofs are not, in themselves, fundamentally different to the standard proofs given by [Kobayashi and Nomizu, 1963]. Rather, the Lie groupoid/Lie algebroid formulation provides a conceptual framework in which the results arise naturally and inevitably. This formulation is scarcely possible if one works solely with principal bundles.

Perhaps more surprisingly, one may prove **[CT–1]** and **[CT–2]** directly and then deduce from them the locally trivial Lie groupoid/Lie algebroid versions of **[Lie–1]** and **[Lie–2]**. Thus connection theory is actually coextensive with the Lie theory of locally trivial Lie groupoids and Lie algebroids.

From a pedagogical point of view, the demonstration that connection theory is a generalization of Lie theory fits it firmly into the broad framework of contemporary mathematics: Lie theory is the study of a functor, and its fundamental results concern the faithfulness and fullness of this functor.

Nonetheless this has not been the place to give a full expository treatment of connection theory: considerations of space and balance mean that Part II contains no significant examples or motivation outside the theory itself.

Looking at connection theory in a broad sense, there have always been three distinct approaches. One may treat it 'globally' in terms of differential forms and exterior calculus; one may treat it 'locally', either by a localization of the differential form approach, or by some form of tensor analysis; finally, one may treat it in terms of path lifting and holonomy. These approaches correspond respectively to the three great cohomology theories which concern differential geometry: the de Rham cohomology, Čech cohomology, and singular cohomology.

The close relationships between these approaches sometimes obscures the fact that they may be treated independently: I demonstrate in the course of Part II that both the global and the local approaches are intrinsic to transitive Lie algebroids and may be developed entirely independently of any underlying Lie groupoid. The place of classical de Rham cohomology is taken by Lie algebroid cohomology, which emerges from a straightforward extension of the calculus of differential forms. On the other hand, considerations involving path–lifting and holonomy require an underlying locally trivial Lie groupoid. The relationships between these three approaches can now be seen as instances of the Lie theory of Lie groupoids and Lie algebroids.

In the writing of Part II, I have had in mind a reader who has met the standard theory of connections in vector bundles, and who does not need to be given geometric motivation for the notion. Nonetheless, all basic definitions are included.

Here now is a brief description of the contents of the individual chapters. **Chapter 5** is an account of the infinitesimal part of the theory of connections in transitive Lie algebroids, both the global calculus and the local description. For the latter, I anticipate Chapter 8 by using a temporary concept of *locally trivial Lie algebroid*; Theorem 8.2.1 shows that all transitive Lie algebroids are locally trivial.

Chapter 6 is primarily concerned with path connections and holonomy. I begin however with proofs of [**Lie–1**], [**Lie–2**] and [**Lie–5**] for locally trivial Lie groupoids, with [**Lie–5**] in §6.1 and [**Lie–1**] and [**Lie–2**] in §6.2. The proofs (given for the first time in [Mackenzie, 1987a]) depend heavily on local triviality, but are for that reason more immediately accessible. Path connections and their holonomy are treated in §6.3. The important §6.4 deduces forms of the Ambrose–Singer and Reduction Theorems of connection theory from [**Lie–1**] and [**Lie–2**]. Using the results of §6.4, §6.5 establishes several 'local constancy' results for transitive Lie algebroids: the kernel of the anchor map is a Lie algebra bundle; morphisms of transitive Lie algebroids are of locally constant rank, and so on.

Chapter 7 is an extended account of the cohomology of Lie algebroids. On a formal level this cohomology is a simple generalization of Chevalley–Eilenberg cohomology of Lie algebras and de Rham cohomology, and has been studied on many occasions; however, to make use of this formalism for smooth structures, it is necessary to ensure at all stages that the constructions respect the underlying vector bundle structure, and this requires the results of §6.5. The account of non–abelian extension theory in §7.3 is (I hope the reader will agree) an elegant demonstration of the geometric content of a purely algebraic formalism: for example the (second) Bianchi identity emerges as the condition that the obstruction class of the coupling (*abstract kernel* in the terminology of Mac Lane [1995]) associated to a transitive Lie algebroid is zero. The results of this section are the foundation for Chapter 8.

§7.4 treats the spectral sequence of a transitive Lie algebroid; this unites the Hochschild–Serre spectral sequence of an extension of Lie algebras with the Leray–Serre spectral sequence of a principal bundle.

Chapter 8 is concerned with the integrability problem for transitive Lie algebroids; that is, with [**Lie–3**]. The main result, Theorem 8.3.9, produces a Čech class associated to any transitive Lie algebroid, the *(cohomological) integrability obstruction.* If $\widetilde{M}$ is the universal cover of the base of the Lie algebroid A, and $Z\widetilde{G}$ is the centre of the universal covering Lie group of the fibre type of the adjoint bundle of A, then the integrability obstruction is in $\check{H}^2(\widetilde{M}, Z\widetilde{G})$. For the Lie algebroid to be integrable, it is not necessary that this class be zero, but that it lie within a discrete subgroup of $Z\widetilde{G}$; equivalently, that it can be sent to zero by the map on cohomology induced by a covering map $\widetilde{G} \to G$, for some G. This class is thus a non–abelian version of the first Chern class of a circle bundle and §8.1 is spent in recalling the various aspects of the first Chern class which are relevant here. The detailed construction of the class is in §8.3. The construction of the integrability obstruction depends crucially on the result 8.2.1 that a transitive Lie algebroid on a contractible base admits a flat connection. This result also provides, in §8.2, a local description of transitive Lie algebroids by families of local Maurer–Cartan forms; this is an infinitesimal analogue of the description of principal bundles by transition functions.

I have not considered the construction of non–integrable transitive Lie algebroids at any length; this, of course, requires techniques outside of Lie groupoid theory. The most natural source is the theory of transversally complete foliations developed by Molino [1988]; the original examples of non–integrable Lie algebroids of Almeida and Molino

[1985] are of this type. A thorough account of the Molino theory and its relation to integrability has been given by Moerdijk and Mrčun [2003].

The extension of [**Lie–1**] to [**Lie–5**] to general Lie groupoids and Lie algebroids is a far more complex matter, and has been the subject of much recent work. Some account of the problems involved and recent achievements are given in the Appendix.

Part I

Lie groupoids (*groupoïdes différentiables*) were introduced by Charles Ehresmann in the 1950s as an alternative to the principal bundle language which he had developed at about the same time as Steenrod. (A detailed history is given in [Ehresmann, 1984, Vol. I].) Ehresmann appears to have been most concerned with higher–order prolongation processes and the notion of a connection of arbitrary order, but it is evident that from the outset he was also concerned with Lie groupoids as holonomy structures in foliation theory.

Lie algebroids were defined by Pradines in the second of a series of short papers [1966, 1967a, 1967b, 1968] outlining a Lie theory for Lie groupoids. The cognate, purely algebraic concept of a Lie pseudoalgebra had already been discussed by a number of authors, and forms of this concept have continued to be introduced since; Pradines' concept of Lie algebroid is unique in positing an underlying structure of smooth vector bundle, and it is this which makes possible the Lie theory of Lie groupoids. Pradines' four short papers were mainly concerned with setting out the theory, which in the case of general Lie groupoids is intimately bound up with foliation theory. This theory lay largely undeveloped for several years.

In [Mackenzie, 1987a] I set out to establish Pradines' Lie theory in rigorous detail for the locally trivial case and, at the same time, to demonstrate the close links with connection theory which have been described above. As well as the restriction to the locally trivial case, [Mackenzie, 1987a] also largely restricted itself to morphisms which are base–preserving; the two restrictions are equally natural when connection theory is the main application.

Arbitrary morphisms of general Lie algebroids are — rather surprisingly — hard to handle; a detailed treatment was given by Philip Higgins and myself [Higgins and Mackenzie, 1990a]. This paper showed that Lie algebroids possess an infinitesimal version of the properties characteristic of general groupoids and introduced a notion of quotient, or descent, for Lie algebroids which had hitherto been lacking.

Part I is concerned with the differentiation processes which relate Lie groupoids and Lie algebroids. **Chapter 1** is concerned with the most basic constructions and fundamental examples for Lie groupoids. Local triviality is treated in full, including the relationship with principal bundles: the fact that this is not a true equivalence has as a consequence that the automorphism groups of a locally trivial Lie groupoid and of a corresponding principal bundle do not correspond. Most of this chapter deals with constructions which resemble processes familiar from Lie group theory, with [**Lie–4**] in §1.5. One exception is the notion of bisection treated in §1.4. Bisections — or 'generalized elements' — correspond to left– and right– translations on a groupoid and are needed later for the adjoint formulas and exponential map. I include the striking description, due to Xu [1995], of the multiplication in a tangent groupoid in terms of bisections.

The algebraic properties of groupoids differ fundamentally from those of groups in that several basic processes may be characterized in terms of morphisms. The first of these is the characterization of actions given in §1.6. From a groupoid action may be constructed an *action groupoid* and a morphism from it to the acting groupoid; these *action morphisms* may be characterized intrinsically and are in bijective correspondence with the actions. This construction enables groupoid actions, and their infinitesimal forms, to be subsumed under the study of the Lie functor for morphisms. This construction goes back to [Ehresmann, 1957]. It was considerably developed by Ronnie Brown who gave a detailed demonstration that, in particular, it provides an algebraic model of covering spaces [Brown, 1988].

The final §1.7 presents the theory of frame groupoids and linear actions of Lie groupoids on vector bundles. Theorem 1.6.23, which shows that stabilizer subgroupoids are, under simple conditions, locally trivial Lie subgroupoids, is applied to tensor structures on vector bundles. This material is central to the connection theory of Part II.

Chapter 2 is chiefly concerned with quotienting processes for Lie groupoids. I begin with the case of vector bundles, which are a simple special case of both Lie groupoids and of Lie algebroids. Here it is well known that the kernel of a surjective morphism $F\colon E \to E'$ over a surjective submersion $f\colon M \to M'$ does not determine F except when the base map is a diffeomorphism; one may say that the 'First

Isomorphism Theorem' breaks down. To restore it, one clearly needs to use the kernel pair $R(f) = \{(y,x) \mid f(y) = f(x)\} \subseteq M \times M$ of f, but it is also necessary to have $R(f)$, regarded as a Lie groupoid on M, acting on E/K, where K is the usual concept of kernel. With the concept of kernel enlarged to incorporate $R(f)$ and the action, one recovers a First Isomorphism Theorem for vector bundles [Higgins and Mackenzie, 1990a].

The class of vector bundle morphisms which is thus characterized by the enlarged concept of kernel, is that where both the map of the total spaces and of the base spaces are surjective submersions. For Lie groupoids the corresponding class of morphisms is the *fibrations*; these arose in the context of set–groupoids as models of Hurewicz fibrations in topology [Brown, 1988], and elsewhere in category theory, and are the broadest simply–defined class of Lie groupoid morphisms for which a First Isomorphism Theorem can be expected. This is the subject of §2.4.

Fibrations also arise in connection with semi–direct products. The general concept of action appropriate to groupoids leads in §2.5 to a very general concept of semi–direct product. It turns out that the natural projection from a general semi–direct product to the acting groupoid is a fibration which is *split* in a specific sense, and conversely, every split fibration defines a semi–direct product.

The long **Chapter 3** contains the basic theory of Lie algebroids and the properties of the Lie functor. For the convenience of readers who know the theory of principal bundles well, and who wish to start directly with Lie algebroids, §3.1 and §3.2 give the construction of the Atiyah sequence of a principal bundle independently of Chapters 1 and 2. The basic definitions and examples of abstract Lie algebroids are in §3.3.

In order to prepare for the Lie algebroids of frame groupoids, §3.4 treats linear vector fields on vector bundles in some detail. A linear vector field on a vector bundle (E, q, M) corresponds to a *derivation* in E — that is, to a first–order differential operator with scalar symbol (named a 'covariant differential operator' in [Mackenzie, 1987a]) — and there are several different formulations, in part developed by myself and Ping Xu [1998], which will be needed throughout the rest of the book. For any Koszul connection ∇ in E, the operators ∇_X, $X \in \mathfrak{X}(M)$, are derivations and some of the results of this section are intrinsic forms of results well known for connections.

The construction of the Lie algebroid of a Lie groupoid and the basic examples are given in §3.5. The two sections which follow, §3.6 and §3.7, treat the exponential map and the adjoint formulas for a general Lie groupoid. The exponential map of a Lie groupoid takes values in (local) bisections rather than in elements and I have endeavoured to make clear how to work in practice with these formulas. §3.6 also contains the calculation of the Lie algebroids of the frame groupoids of a vector bundle with a tensor structure; these results are basic to Part II.

Chapter 4 is concerned with infinitesimal versions of the results of Chapter 2. Whereas most constructions for Lie groupoids differentiate readily enough to Lie algebroids, if can be unexpectedly difficult to give an abstract form of these results for arbitrary Lie algebroids, independently of any integrability assumptions.

Although the crucial difficulty is with the concept of morphism, I begin with the case of actions of Lie algebroids. A natural abstract definition presents itself (4.1.1) in this case. Given an action of a Lie algebroid A with base M on a smooth map $f\colon M' \to M$, there is an *action Lie algebroid* $A \lessdot f$ on M' and the natural projection from $A \lessdot f$ to A enjoys an infinitesimal form of the property characteristic of action morphisms of Lie groupoids. I take this infinitesimal property as defining an *action morphism of Lie algebroids* and establish an equivalence between action morphisms of Lie algebroids and Lie algebroid actions, which commutes with the Lie functor and the corresponding equivalence on the groupoid level.

The case of actions provides a paradigm for the other results of this chapter: (i) characterize a groupoid construction in terms of a class of groupoid morphisms or maps; (ii) apply the Lie functor to this class and characterize the resulting Lie algebroid morphisms abstractly, without reference to the differentiation process by which they were obtained; (iii) prove that this class of Lie algebroid morphisms corresponds to a Lie algebroid construction analogous to the original groupoid construction. This method, developed in [Higgins and Mackenzie, 1990a], avoids a considerable amount of unilluminating computation.

The general concept of morphism for Lie algebroids is dealt with in §4.3, following necessary preliminaries on direct products and pullbacks in §4.2. The Lie functor set out in §3.5 is straightforward; what is at issue in §4.3 is the abstract definition. The definition given here includes maps arising from the Lie functor, base–preserving morphisms, pullback projections, and action morphisms. It has been shown [Mackenzie and Xu, 2000] to pass the final test, that such a morphism between the Lie algebroids of Lie groupoids integrates suitably to a morphism of Lie groupoids.

Using this concept of morphism and the paradigm described above, I give concepts of fibration and quotient (§4.4) and general semi–direct product and split fibration (§4.5) for Lie algebroids. This, like most of the material of this Chapter, is based upon work of Philip Higgins and myself [1990a].

The general concepts of morphism for Lie groupoids and Lie algebroids have been crucial for much subsequent work: the criterion 12.2.1 for a Lie bialgebroid requires a general concept of Lie algebroid morphism, as does the development of a Lie theory for double groupoids.

Preface

This book was originally intended to be largely disjoint from my earlier book *Lie groupoids and Lie algebroids in differential geometry*, [Mackenzie, 1987a], written in the period from 1980 to mid 1985 and published in the London Mathematical Society Lecture Note series in 1987. However I have included, in Part II of the present book, the central chapters of the earlier book on transitive Lie algebroids, their cohomology and connection theory and their integrability. Despite the dramatic results since 2000 on the integrability problem for general Lie algebroids, and work on more sophisticated cohomology theories, it seems to me well worth while to continue to treat the case of locally trivial Lie groupoids and transitive Lie algebroids independently of the general theory. As I have noted already, the systematic use of Lie groupoids and abstract Lie algebroids provides a thoroughgoing reformulation of standard connection theory, and is likely to retain its own character independent of the more general results. This material has in some cases been rewritten and in others left almost unchanged, though typos and obscurities have, I hope, always been caught.

The earlier book was intended to be readable without a detailed prior knowledge of connection theory, and certainly without any acquaintance with groupoids or Lie algebroids, and was consequently leisurely in pace. I feel it is no longer necessary to argue throughout the book for the importance of groupoids in differential geometry, and I have now also assumed that readers have a basic knowledge of connection theory and principal bundles, as well as the standard processes of manifolds, vector bundles and Lie groups.

Readers familiar with the earlier book may note the following broad changes:

- Almost all of the material on topological groupoids in Chapter II of [Mackenzie, 1987a] has been omitted. That part which is relevant to the theory of Lie groupoids has been retained, but is now given only in terms of smooth structures. Most of the remaining material dealt with phenomena which cannot occur for Lie groupoids, and perhaps it will be of interest in some future study of topological groupoids. However, I do not pursue that possibility here.

- The material on locally trivial Lie groupoids and transitive Lie algebroids has been thoroughly revised and developed, though without altering the nature of the approach in any fundamental way. I have added a brief expository section, §5.2, on the connection theory of vector bundles. The aim has been to make the account of transitive Lie algebroids technically self–contained, but §5.2 is not a complete introduction to connections in vector bundles.

  I have expanded the treatment of the integrability obstruction so as to give a more complete account of this subject, and elsewhere I have compressed the earlier account, where it seemed too detailed.

- Historical material is now placed in Notes at the end of each chapter, together with comments on other approaches, omitted material, and so on. These are not comprehensive historical surveys, but are as complete as I could make them in a reasonable time.

* * * * *

This book contains several subbooks, which may be read independently of the rest of the text:

• Readers interested in the locally trivial/transitive theory of Part II need read only Chapter 1 and Chapter 3 as preparation.

• Those interested in Lie algebroids but not the underlying groupoid theory may concentrate on §2.1, §3.1 to §3.4, and Chapter 4. For a reader who already knows the standard theory of connections in vector bundles, Chapter 5 and Chapter 7 will then be accessible.

• Those wanting an introduction to double vector bundles may read §3.4 and then sections 9.1 to 9.6.

• Those interested mainly in Part III may omit Part II. Much of Part III requires only Chapter 1 and Chapter 3, but reference back to Chapter 2 and Chapter 4 will be needed for some results.

I hope that this book will be used both as a reference, and as a source for those learning the subject. With the first aim in view, topics have been placed, with one or two short exceptions, in their logical position, rather than where they are first needed, and there is extensive cross–referencing. On the other hand, I have gone to some lengths to explain the general approach and most individual transitions, and I hope that readers new to the subject will not lack for signposts.

* * * * *

This book is by no means an account of everything that is known on the subject of Lie groupoids and Lie algebroids; topics which are not covered include the general theory of holonomy and monodromy, the several recent very general integrability results, and multiple Lie theory. Some comments and references on topics which have been omitted are collected in the Appendix.

Apart from the universal constraints of space and availability under which everyone works, my defence for omissions is that I have set out to present three main strands of the theory — with which I have been most involved — in a philosophically coherent and motivated way. Certainly the subject is rich enough to be developed from other perspectives than those adopted here.

Listed below are monographs and survey articles that contain substantial alternative treatments of aspects of groupoid theory or of Lie algebroids. Full references are given in the bibliography.

I. Moerdijk and J. Mrčun, *Introduction to Foliations and Lie groupoids.* Cambridge University Press, 2003.

A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, American Mathematical Society, 1999.

A. Paterson, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Birkhäuser Boston Inc. 1999.

N. P. Landsman, *Mathematical Topics between Classical and Quantum Mechanics*, Springer–Verlag, 1998.

A. Weinstein, 'Groupoids: Unifying internal and external symmetry', *Notices of the Amer. Math. Soc.*, 1996.

K. C. H. Mackenzie, "Lie algebroids and Lie pseudoalgebras", *Bull. London Math. Soc.*, 1995.

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Birkhäuser, 1994.

M. V. Karasëv and V. P. Maslov, *Nonlinear Poisson brackets, Geometry and Quantization*, American Mathematical Society, 1993.

R. Brown, *Topology: a Geometric Account of General Topology, Homotopy Types and the Fundamental Groupoid*, Ellis Horwood, 1988.

R. Brown, "From groups to groupoids: a brief survey", *Bull. London Math. Soc.*, 1987.

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TERMINOLOGY AND NOTATION

⋆ All manifolds are pure, Hausdorff, finite–dimensional and second–countable, except where explicitly stated otherwise.

⋆ The expression 'Lie groupoid' in this book does not include a local triviality condition. We take Lie groupoids in the same sense (except as regards concepts of manifold; see the previous point) as the differentiable groupoids of [Pradines, 1966] and [Mackenzie, 1987a], and the smooth groupoids of [Connes, 1994].

⋆ For a vector bundle E, following [Kosmann–Schwarzbach and Mackenzie, 2002], I have replaced the term 'covariant differential operator on E' by 'derivation on E' or 'derivative endomorphism of ΓE' and replaced the notation $\mathrm{CDO}(E)$ by $\mathfrak{D}(E)$.

⋆ The semi–direct product notation $\ltimes$ has in the past been frequently used in three distinct senses, both for Lie groupoids and Lie algebroids: for action (or transformation) structures, for semi–direct products in which the base manifold remains fixed, and for 'general semi–direct products', combining the first two constructions. This could be seriously confusing: for example one was led to write $T^*G \cong G \ltimes \mathfrak{g}^*$ and $TG \cong G \ltimes \mathfrak{g}$, for a Lie group G, where in the first case $G \ltimes \mathfrak{g}^*$ was the action Lie groupoid and in the second $G \ltimes \mathfrak{g}$ was the usual semi–direct product group. I denote these three senses by, respectively, the symbols $\lessdot$, $\ltimes$, $\boxminus$.

⋆ The exterior powers of a vector bundle A are denoted $\Lambda^k(A)$ and those of its dual $\Lambda^k(A^*)$. Since the symbol Ω is used for a locally trivial Lie groupoid, the p–forms on a manifold M are denoted not by $\Omega^p(M)$ but by $\Omega^{\mathsf{p}}(M)$. The graded ring of all differential forms is $\Omega^\bullet(M)$. The module of vector fields is $\mathcal{X}(M)$ and the Schouten algebra of all multivector fields is $\mathcal{X}^\bullet(M)$.