A UNIFIED APPROACH TO POISSON REDUCTION *

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Abstract

Given any Poisson action $G \times P \to P$ of a Poisson–Lie group G we construct an object $\Omega = T^*G * T^*P$ which has both a Lie groupoid structure and a Lie algebroid structure and which is a half–integrated form of the matched pair of Lie algebroids which J.–H. Lu associated to a Poisson action in her development of Drinfeld's classification of Poisson homogeneous spaces. We use Ω to give a general reduction procedure for Poisson group actions, which applies in cases where a moment map in the usual sense does not exist. The same method may be applied to actions of symplectic groupoids and, most generally, to actions of Poisson groupoids.

1 Introduction

Consider moment maps in the most classical case: G is a Lie group, (P, ω) is a symplectic manifold, and $G \times P \to P$ is an action which preserves the symplectic structure and so induces an infinitesimal action $\mathfrak{g} \to \mathcal{X}(P),\ X \longmapsto X^\dagger$ in which every fundamental vector field is locally Hamiltonian. The action is Hamiltonian if there is a map $\mathfrak{g} \to C^\infty(P),\ X \longmapsto u_X$, necessarily $\mathbb{R}$–linear, such that $X^\dagger = (du_X)^\#$ for all $X \in \mathfrak{g}$, where $^\#$ denotes the canonical map $T^*P \to TP$. The corresponding map $\mu\colon\ P \to \mathfrak{g}^*$ is then the moment map of the action.

Now consider this differently. Regarding the infinitesimal action as $P \times \mathfrak{g} \to TP$, take the dual and drop the base variable so as to obtain a map $\mathfrak{p}\colon\ T^*P \to \mathfrak{g}^*$. Using the canonical isomorphism $T^*P \cong TP$, this gives a closed $\mathfrak{g}^*$–valued 1–form $\theta\colon\ TP \to \mathfrak{g}^*$. If P is simply–connected and $x_0 \in P$ is any reference point, there is a smooth map $\mu\colon\ P \to \mathfrak{g}^*$ such that $d\mu = \theta$. This yields, of course, the same maps μ as before.

Here T^*P is a Lie algebroid under the Poisson bracket of 1–forms and the canonical isomorphism $T^*P \cong TP$ is an isomorphism of Lie algebroids to the standard tangent bundle,

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which is the Lie algebroid of the pair groupoid $P \times P$. Regarded like this, the fact that θ is closed means that it is a Lie algebroid morphism from TP to the abelian Lie algebra $\mathfrak{g}^*$ and when P is simply–connected this may be integrated to a Lie groupoid morphism $\mathfrak{P}\colon P \times P \to \mathfrak{g}^*$ which is necessarily of the form $\mathfrak{P}(y,x) = \mu(y) - \mu(x)$ where μ is obtained from $\mathfrak{P}$ by choosing any $x_0 \in P$ and defining $\mu(x) = \mathfrak{P}(x, x_0)$. The existence of moment maps has thus been reduced to the integration of Lie algebroid morphisms.

This viewpoint can be applied more generally. Consider next a symplectic action of a symplectic groupoid as defined by Mikami and Weinstein [21]: given a symplectic groupoid $\Sigma \Longrightarrow M$ and a symplectic manifold P, provided with a smooth map $f\colon P \to M$, an action of Σ on P is a symplectic action if the graph is a canonical relation. Again we take the infinitesimal action $\Gamma A\Sigma \to \mathcal{X}(P)$ and dualize it to $\mathfrak{p}\colon T^*P \to A^*\Sigma$. Again it may be proved [30] that this is a Lie algebroid morphism (with base map $f\colon P \to M$). Composing with the canonical isomorphisms $T^*P \cong TP$ and $A^*\Sigma \cong TM$ we can in fact identify $\mathfrak{p}$ with $T(f)\colon TP \to TM$. This integrates (without simple connectivity hypotheses) to $f \times f\colon P \times P \to M \times M$ which may in turn be identified with f, which [21] shows may be regarded as the moment(um) map of the action.

Next consider a Poisson action $G \times P \to P$ of a Poisson group on a Poisson manifold. Again taking the dual of the infinitesimal action, we obtain a Lie algebroid morphism $\mathfrak{p}\colon T^*P \to \mathfrak{g}^*$. If P integrates to an α–simply connected symplectic groupoid $\Pi \Longrightarrow P$ and G^* is a dual group (integrating the Lie algebra $\mathfrak{g}^*$) then $\mathfrak{p}$ may be integrated to a Lie groupoid morphism $\mathfrak{P}\colon \Pi \to G^*$. If a moment map $\mu\colon P \to G^*$ in the sense of Lu [8] exists, then $\mathfrak{P}$ is the composite of the anchor $\Pi \to P \times P$ with the map $P \times P \to G^*$, $(u_2, u_1) \longmapsto \mu(u_2)\mu(u_1)^{-1}$, derived from μ [30, §6]; see §4.

Classical Marsden–Weinstein reduction for a Hamiltonian action $G \times P \to P$ with moment map $\mu\colon P \to \mathfrak{g}^*$ calculates the symplectic leaves of the reduced Poisson space P/G; even when P/G is not a manifold, the leaves of the abstract Poisson structure are (under well–known conditions) the symplectic manifolds $\mu^{-1}(\theta)/G_\theta$ for $\theta \in \mathfrak{g}^*$. For a general Poisson manifold P, the characteristic foliation will usually not be regular, but one may still be able to obtain a symplectic groupoid $\Pi \Longrightarrow P$; indeed a local symplectic groupoid Π with base P always exists [6], [7], [24], [25]. Such a Π represents a desingularization of the characteristic foliation of P: the Lie algebroid $A\Pi$ of the Lie groupoid structure is canonically isomorphic to the cotangent Lie algebroid T^*P of the Poisson structure on the base; more strongly, the transitivity orbits of the groupoid structure are precisely the symplectic leaves of P. Thus if we are given a Poisson group action $G \times P \to P$ and a symplectic groupoid $\Pi \Longrightarrow P$, a satisfactory description of the Poisson structure on P/G (assuming it exists as a manifold) is given by producing a symplectic groupoid for P/G. This was done for symplectic groupoid actions by Ping Xu in [29]; our main purpose in this paper is to show how a similar construction applies to Poisson group actions.

In fact all the cases mentioned above are special cases of a reduction process for actions of Poisson groupoids or Lie bialgebroids; this most general construction will be dealt with in a subsequent paper.

Basic to what follows is the concept of $\mathcal{LA}$-groupoid which was introduced in [13, §4] as an object intermediate between the Lie bialgebra of a Poisson group and its symplectic double groupoid. We recall this notion in §2. In §3 we show that the matched pair of Lie algebroids which Lu [9] constructs in order to develop Drinfel'd's classification of Poisson homogeneous spaces [3] arises from a certain $\mathcal{LA}$-groupoid in a natural way; this is the basis of our approach to reduction in §4. We assume that the reader is familiar with Lie groupoids, Lie algebroids, and symplectic groupoids: see [2], [23] or [14].

2 Poisson Lie groups and vacant $\mathcal{LA}$-groupoids

We regard a Poisson structure on a manifold P as a bracket on the module $C^\infty(P)$ of smooth functions on P with respect to which $C^\infty(P)$ is an $\mathbb{R}$–Lie algebra and which is a derivation in each variable separately:

$$\{u, vw\} = v\{u, w\} + w\{u, v\}$$

for $u, v, w \in C^\infty(P)$. Associated with this is the 2–tensor $\pi\colon TP \oplus TP \to P \times \mathbb{R}$ defined by

$$\pi(u\,dv, u'\,dv') = uu'\{v, v'\}$$

for $u, v \in C^\infty(P)$; this is the *Poisson tensor*. We mostly work with the map $\pi^\#\colon T^*P \to TP$ defined by

$$\langle \pi, \theta_1 \wedge \theta_2 \rangle = \langle \theta_2, \pi^\#(\theta_1) \rangle;$$

we call this the *Poisson anchor* and often write $\theta^\# = \pi^\#(\theta)$. The bracket of 1–forms on P is now given by

$$[\theta_1, \theta_2] = \mathcal{L}_{\theta_1^\#}(\theta_2) - \mathcal{L}_{\theta_2^\#}(\theta_1) - d(\pi(\theta_1.\theta_2)) \qquad (1)$$

where $\theta_1, \theta_2 \in \Lambda^1(P)$. The bracket of 1–forms and the map $\pi^\#\colon T^*P \to TP$ define the structure of a Lie algebroid on T^*P in the following sense. See [1], [23] and, for a historical survey, [5].

Definition 2.1 *Let M be a manifold. A* Lie algebroid *on M is a vector bundle (A, q, M) together with a vector bundle map $a\colon A \to TM$ over M, called the* anchor *of A, and a bracket $[\ ,\ ]\colon \Gamma A \times \Gamma A \to \Gamma A$ which is $\mathbb{R}$-bilinear and alternating, satisfies the Jacobi identity, and is such that for all $X, Y \in \Gamma A$, $u \in C^\infty(M)$.*

$$a([X, Y]) = [a(X), a(Y)] \qquad \textit{and} \qquad [X, uY] = u[X, Y] + a(X)(u)Y.$$

For background on Lie algebroids see [14] and references given there. The concept of morphism for Lie algebroids requires care and will be needed throughout the paper. There are various formulations; the one most useful here is from [4]. The notation $f^!A$ means the pullback vector bundle; we reserve asterisks for duals.

Definition 2.2 *Let A' and A be Lie algebroids over M' and M, and let $\varphi\colon A' \to A$ be a vector bundle morphism over $f\colon M' \to M$. Denote the induced map $A' \to f^!A$ by $\varphi^!$.*

Then (φ, f) is a morphism of Lie algebroids *if $a \circ \varphi = T(f) \circ a'$ and if, given $X', Y' \in \Gamma A'$ for which $\varphi^!(X') = \sum_i u_i' \otimes X_i$ and $\varphi^!(Y') = \sum_j v_j' \otimes Y_j$, we have*

$$\varphi^!([X', Y']) = \sum_{i,j} u_i' v_j' \otimes [X_i, Y_j] + \sum_j a'(X')(v_j') \otimes Y_j - \sum_i a'(Y')(u_i') \otimes X_i.$$

The morphism (φ, f) is an action morphism *if $\varphi^!\colon A' \to f^!A$ is an isomorphism of vector bundles.*

Definition 2.3 *Let* A *be a Lie algebroid on* M *and let* $f\colon M' \to M$ *be a smooth map. An* action of A on f, *or* on M', *is an* $\mathbb{R}$*–linear map* $\Gamma A \to \mathcal{X}(M')$, $X \longmapsto X^{\S}$, *such that, for* $X, Y \in \Gamma A$, $u \in C^\infty(M)$,

$$[X,Y]^{\S} = [X^{\S}, Y^{\S}], \qquad (uX)^{\S} = (u\circ f)X^{\S}, \qquad X^{\S} \textit{ projects under } f \textit{ to } a(X).$$

An action can be lifted to a map $f^!A \to TM'$ by $(m', X) \longmapsto X^{\S}(m')$; denote this by a'. Then a' is the anchor for a Lie algebroid structure on $f^!A$. The bracket is defined for pullback sections $\overline{X}$ by

$$[\overline{X}, \overline{Y}] = \overline{[X,Y]}$$

and extended to all of $\Gamma(f^!A)$ by the Leibniz rule in 2.1 and additivity. With this structure we denote $f^!A$ by $A \lessdot f$ or $A \lessdot M'$; it is the *action Lie algebroid* corresponding to the given action. The canonical map $A \lessdot f \to A$ is an action morphism. (Action structures have usually been denoted by $\ltimes$; we reserve this symbol for semi–direct products in the standard sense.)

Conversely, given an action morphism as in 2.2, the formula $X^{\S} = a'(\overline{X})$ defines an action of A on f, where $\overline{X}$ is the pullback section corresponding to X, the unique section of A' such that $\varphi \circ \overline{X} = X \circ f$. For more details on these constructions see [4].

We can now consider the construction which is fundamental to this paper. Let G be a Lie group. The group structure induces on T^*G a Lie groupoid structure over base $\mathfrak{g}^*$. For $\theta \in T^*_gG$, the source and the target are

$$\beta(\theta) = \theta \circ T(R_g), \qquad \alpha(\theta) = \theta \circ T(L_g),$$

where R_g and L_g are the right and left translations for G. For θ as above and $\varphi \in T^*_hG$ the multiplication is

$$\varphi\theta = \varphi \circ T(R_{g^{-1}}) = \theta \circ T(L_{h^{-1}}).$$

This is the cotangent groupoid of G. See [1] and, for more background, [14].

Definition 2.4 *Let* G *be a Lie group and let* π *be a Poisson structure on* G. *Then* (G,π) *is a* Poisson Lie group *if* $\pi^\#\colon T^*G \to TG$ *is a morphism of Lie groupoids from the cotangent groupoid to the usual tangent group.*

This is the definition used for Poisson groupoids in [18], applied to the group case; see also [13, 4.12].

The morphism condition is equivalent to the condition that for $\varphi \in T^*_gG$, $\psi \in T^*_hG$ with $\varphi \circ T(L_g) = \psi \circ T(R_h)$ we have

$$\pi^\#(\varphi \circ T(R_{h^{-1}})) = \pi^\#(\varphi) \bullet \pi^\#(\psi) = T(R_h)(\pi^\#(\varphi)) + T(L_g)(\pi^\#(\psi)). \qquad (2)$$

Here $\bullet$ denotes the multiplication in the usual tangent group TG.

Using the right trivializations $T^*G \cong G \times \mathfrak{g}^*$ and $TG \cong G \times \mathfrak{g}$ of the bundles we can define

$$\pi^R\colon G \to \mathrm{Lin}(\mathfrak{g}^*, \mathfrak{g}) \quad \text{by} \quad \pi^R(g)(\theta) = T(R_{g^{-1}})(\pi^\#(\theta \circ T(R_{g^{-1}})).$$

Then (2) implies that, for $\theta \in \mathfrak{g}^*$ and any $g, h \in G$,

$$\pi^R(gh)(\theta) = \pi^R(g)(\theta) + \mathrm{Ad}_g(\pi^R(h)(\theta)$$

or, briefly, $\pi^R(gh) = \pi^R(g) + \mathrm{Ad}_g\,\pi^R(h)$, where $\mathrm{Ad}_g\colon \mathrm{Lin}(\mathfrak{g}^*, \mathfrak{g}) \to \mathrm{Lin}(\mathfrak{g}^*, \mathfrak{g})$ maps θ to $\mathrm{Ad}_g \circ\theta \circ \mathrm{Ad}_g$. This argument is easily reversed and shows the equivalence of 2.4 with the usual definition.

We take the Lie algebra structure on $\mathfrak{g}^*$ to be defined by

$$[\overleftarrow{\theta_1}, \overleftarrow{\theta_2}] = \overleftarrow{[\theta_1, \theta_2]} \tag{3}$$

where, given $\theta \in \mathfrak{g}^*$, the 1–form $\overleftarrow{\theta}$ on G is defined by

$$\overleftarrow{\theta}(g) = \theta \circ T(L_{g^{-1}}).$$

Then $\overleftarrow{\theta}$ is *left–invariant* in the sense that for all $X \in \mathfrak{g}$ the function $\langle \overleftarrow{\theta}, \overleftarrow{X}\rangle$ is constant; conversely every such 1–form on G is $\overleftarrow{\theta}$ for some θ. For the proof that the bracket of invariant 1–forms is invariant, see [27].

This construction makes it plain that $\widetilde{\alpha}\colon T^*G \to \mathfrak{g}^*$ is a morphism of Lie algebroids. Given $\theta \in \mathfrak{g}^*$, the 1–form $\overleftarrow{\theta}$ is the pullback section corresponding to θ, and (3) now shows both that $\widetilde{\alpha}$ is a morphism and that it is an action morphism of Lie algebroids. That is, $\theta^\S = \pi^\#(\overleftarrow{\theta})$ defines an action of $\mathfrak{g}^*$ on G, and the pullback $\widetilde{\alpha}^!\colon T^*G \to G \times \mathfrak{g}^*$ is an isomorphism of Lie algebroids to the action Lie algebroid $\mathfrak{g}^* \lessdot G$.

The map $\widetilde{\alpha}^!$ is precisely the left trivialization $T^*G \to G \times \mathfrak{g}^*$, which we denote by Δ. We can also regard Δ as being induced by the cotangent projection $c\colon T^*G \to G$, which is a morphism of groupoids and an action morphism.

Definition 2.5 *A morphism of groupoids $F\colon G' \to G$ over $f\colon M' \to M$ is an* action morphism of groupoids *if the induced map $F^!\colon G' \to f^!G$, $g' \longmapsto (\alpha'(g'), F(g'))$, is a diffeomorphism.*

Here $f^!G$ is the pullback manifold $M' * G$ of f and α. Given an action morphism, the associated action of G on f is $gm' = \beta'((F^!)^{-1}(m', g))$. Given an action of G on f, the associated *action groupoid* is the manifold $f^!G$ with source $\alpha'(m', g) = m'$, target $\beta'(m', g) = gm'$, and multiplication $(gm', h)(m', g) = (gm', hg)$.

Thus the fact that $c^! = \Delta$ is a diffeomorphism means that c is an action morphism of groupoids; the induced action of G on $\mathfrak{g}^*$ is the standard (left) coadjoint action. Δ is thus also an isomorphism of Lie groupoids from T^*G to the action groupoid $G \lessdot \mathfrak{g}^*$.

Since $\pi^\#$ is a groupoid morphism by assumption, it follows that $\pi^\# \circ \Delta^{-1}\colon G \lessdot \mathfrak{g}^* \to TG$, $(g, \theta) \longmapsto \theta^\S(g)$, is a groupoid morphism; that is,

$$\theta^\S(hg) = \varphi^\S(h) \bullet \theta^\S(g), \tag{4}$$

where $\varphi = \mathrm{Ad}_g^*\,\theta$. Equivalently,

$$\theta^\S(hg) = T(R_g)((\mathrm{Ad}_g^*\theta)^\S(h)) + T(L_h)((\theta)^\S(g)). \tag{5}$$

This is the usual *twisted multiplicativity equation* for the dressing transformations; see, for example, [11].

It is important to notice that $\widetilde{\beta}$ is also a morphism of Lie algebroids, with respect to the same structures. This follows immediately from the next result.

Proposition 2.6 *The groupoid inversion $T^*G \to T^*G$ is an isomorphism of Lie algebroids.*

PROOF. Denote the groupoid inversion $T^*G \to T^*G$ by j and denote the inversion $G \to G$ by j_0. That $\pi^\# \circ j = T(j_0) \circ \pi^\#$ is part of the condition that $\pi^\#$ be a groupoid morphism. Since the bracket in T^*G is determined by $\pi^\#$, the result should in principle follow directly. However the following intermediate steps are useful:

For $\varphi \in \Lambda^1(G)$ denote by $j_*(\varphi)$ the transported section $(j_*(\varphi))(g) = j(\varphi(g^{-1}))$. Then

$$\pi(j_*(\varphi_1), j_*(\varphi_2)) = -\pi(\varphi_1, \varphi_2) \circ j_0. \tag{6}$$

Using this, it follows that for any $\xi \in \mathcal{X}(G)$,

$$L_{j_{0*}(\xi)}(j_*(\varphi)) = j_*(L_\xi(\varphi)), \tag{7}$$

where $j_{0*}(\xi)$ is the usual transport of the vector field. Finally, note that the pullback across j_0 of a 1–form φ and the inversion of φ are related by

$$j_0^*(\varphi) = -j_*(\varphi). \tag{8}$$

∎

It now follows that $\widetilde{\beta} \circ \Delta^{-1}\colon \mathfrak{g}^* \lessdot G \to \mathfrak{g}^*$, $(g, \theta) \longmapsto \mathrm{Ad}_g^*\theta$, is a morphism of Lie algebroids. Denote it temporarily by Φ.

Morphisms of Lie algebroids in which the target is a Lie algebra admit a simpler characterization than in 2.2. Let $\mathfrak{h}$ be a Lie algebra and let A be a Lie algebroid on M. Then for a vector bundle map $\Phi\colon A \to \mathfrak{h}$ to be a Lie algebroid morphism, the anchor condition is vacuous and the bracket condition reduces to

$$\mathcal{L}_{a(X)}(\Phi^!(Y)) - \mathcal{L}_{a(Y)}(\Phi^!(X)) - \Phi^!([X,Y]) = 0$$

where $\Phi^!(Y)$ is regarded as a map $M \to \mathfrak{h}$. See [30, 5.1].

Thus in the present case we have, for any two maps $\theta_1, \theta_2\colon G \to \mathfrak{g}^*$,

$$\mathcal{L}_{\theta_1^\S}(\theta_2^{\mathrm{Ad}}) - \mathcal{L}_{\theta_2^\S}(\theta_1^{\mathrm{Ad}}) - [\theta_1, \theta_2]^{\mathrm{Ad}} = 0, \tag{9}$$

where $\theta^{\mathrm{Ad}}\colon G \to \mathfrak{g}^*$ denotes the map $g \longmapsto \mathrm{Ad}_g^*(\theta(g))$.

The method of 2.6 also proves the next result. For the pullback of two Lie algebroids see [4].

Proposition 2.7 *The groupoid multiplication $T^*G * T^*G \to T^*G$ is a Lie algebroid morphism, where the domain is the pullback Lie algebroid of $\widetilde{\alpha}\colon T^*G \to \mathfrak{g}^*$ and $\widetilde{\beta}\colon T^*G \to \mathfrak{g}^*$.*

Thus the two structures on the cotangent of a Poisson Lie group are compatible in a very strong sense: the structure maps of the groupoid structure are all morphisms of Lie algebroids; that is, T^*G is a groupoid object in the category of Lie algebroids. Notice too that the groupoid structure on $T^*G \rightrightarrows \mathfrak{g}^*$ arises entirely from the Lie group structure on G whereas the Lie algebroid structure on $T^*G \to G$ arises entirely from the Poisson structure.

One feels that a Poisson Lie group G should be a group object in the Poisson category, but this is not strictly true: the group inversion is not Poisson, but anti–Poisson, and the right and left translations are generally neither Poisson nor anti–Poisson. The contravariant nature of the cotangent functor absorbs these phenomena, and T^*G is a genuine Lie groupoid object in the category of Lie algebroids; the groupoid inversion is a Lie algebroid morphism by 2.6 and it follows from 2.7 that the right and left groupoid translations in T^*G are Lie algebroid morphisms.

Lastly, we know from (2) that $\pi^\#$ has rank zero at $1 \in G$, so the kernel of $\pi_1^\#$ is precisely $\mathfrak{g}^*$, with its Lie algebra structure obtained by linearization of π. As with any Lie algebroid at a point where the anchor is zero, $\mathfrak{g}^* \to \{\cdot\}$ is a Lie subalgebroid of $T^*G \to G$. Now the composite of this inclusion with $\widetilde{\alpha}$ is the identity map $\mathfrak{g}^* \to \mathfrak{g}^*$ and since the inclusion and $\widetilde{\alpha}$ are Lie algebroid morphisms, it follows that the identity map is also. Thus the Lie algebra structure on $\mathfrak{g}^*$ obtained from (3) coincides with the linearized structure from π.

We have shown that the cotangent of a Poisson Lie group is an $\mathcal{LA}$-groupoid as defined as follows.

Definition 2.8 [13, p.212] *An* $\mathcal{LA}$-groupoid *is a system* $(\Omega; G, A; M)$ *in which* Ω *has both a Lie algebroid structure over base* G*, and a Lie groupoid structure over base* A*, where* G *is a Lie groupoid on* M *and* A *is a Lie algebroid on* M*, such that the groupoid structure maps for* Ω *(source* $\widetilde{\alpha}$*, target* $\widetilde{\beta}$*, identity map* $\widetilde{1}$*, multiplication, inversion) are Lie algebroid morphisms. It is* vacant *if the map* $\Delta = (\widetilde{q}, \widetilde{\alpha})\colon \Omega \to G \times_M A$ *is a diffeomorphism.*

Denote the Lie algebroid anchors by $\widetilde{a}\colon \Omega \to G$ and $a\colon A \to TM$ and the bundle projections by $\widetilde{q}$ and q. We visualize $\mathcal{LA}$-groupoids as in Figure 1. We now show that the

$$\begin{array}{ccc} \Omega & \overset{\widetilde{\alpha},\widetilde{\beta}}{\Longrightarrow} & A \\ {\scriptstyle \widetilde{q}}\downarrow & & \downarrow{\scriptstyle q} \\ H & \underset{\alpha,\beta}{\Longrightarrow} & M \end{array} \qquad\qquad \begin{array}{ccc} T^*G & \Longrightarrow & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ G & \Longrightarrow & \{\cdot\} \end{array}$$

(a) (b)

Figure 1:

description of a Poisson Lie group in terms of the two actions — the coadjoint action and the dressing transformation action — extends to any vacant $\mathcal{LA}$-groupoid.

By assumption, $\widetilde{\alpha}\colon \Omega \to A$ is a Lie algebroid morphism over $\alpha\colon G \to M$. Because of the vacancy condition it is an action morphism, with corresponding action

$$X^\S = \widetilde{a}(\overline{X}) \in \mathcal{X}(G) \tag{10}$$

where $\overline{X}$ is the unique section of Ω which projects to $X \in \Gamma A$ under $\widetilde{\alpha}$. Further, $\Delta = \widetilde{\alpha}^!\colon \Omega \to A \lessdot G$ is an isomorphism of Lie algebroids.

Next, $\widetilde{q}\colon \Omega \to G$ is an action morphism of Lie groupoids over $q\colon A \to M$. Accordingly, $\Delta = \widetilde{q}^!$ is an isomorphism of Lie groupoids $\Omega \to G \lessdot A$ where the action of G on A is

$$gX = \widetilde{\beta}(\Delta^{-1}(g, X)). \tag{11}$$

Now the composite $\widetilde{a}\circ\Delta^{-1}\colon G \lessdot A \to TG,\ (g,X) \longmapsto X^{\S}(g)$, is a morphism of Lie groupoids and so we have

$$X^{\S}(hg) = (gX)^{\S}(h) \bullet X^{\S}(g) \tag{12}$$

for $X \in A$ and $h, g \in G$ with $q(X) = \alpha(g)$ and $\alpha h = \beta g$.

The multiplication $\bullet$ on the RHS is the multiplication in the tangent groupoid $TG \Longrightarrow TM$ obtained by applying the tangent functor to the structure maps of $G \Longrightarrow M$. When G is a group, TG is the usual semi–direct product $G \ltimes \mathfrak{g}$ with respect to the adjoint action, but for general Lie groupoids there is no comparable description. For an explicit formula in the general case see [30, 2.6]. From (12) it follows that $X^{\S}(1_m) = T(1)(a(X))$ for $X \in A_m,\ m \in M$.

Likewise, $\widetilde{\beta}\circ\Delta^{-1}\colon A \lessdot G \to A,\ (g,X) \longmapsto gX$, is a morphism of Lie algebroids over β. This is another action morphism and so the morphism conditions of 2.2 reduce to anchor preservation and the preservation of projectability under brackets. The anchor preservation condition is that

$$T(\beta)(X^{\S}(g)) = a(gX), \tag{13}$$

for $X \in A$ and $g \in G$ with $qX = \alpha g$. Given $X \in \Gamma A$, the unique section $\widetilde{X}$ which projects under $\widetilde{\beta}\circ\Delta^{-1}$ to X is given by

$$\widetilde{X}(g) = (g, g^{-1}X(\beta g)).$$

The bracket condition is accordingly that, for all $X, Y \in \Gamma A$,

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]}, \tag{14}$$

where the bracket on the RHS is for the action Lie algebroid $A \lessdot G$.

We have proved half of the following result; for the other half see [13].

Theorem 2.9 [13, 4.9] *In a vacant $\mathcal{LA}$-groupoid the actions* (10) *and* (11) *satisfy the three compatibility conditions* (12), (13) *and* (14).

Conversely, given a Lie algebroid A and a Lie groupoid G, an action of G on the vector bundle A, and an action of A on $\alpha\colon G \to M$, satisfying the conditions (12), (13), (14), *there is a unique vacant $\mathcal{LA}$-groupoid which induces them.*

Equation (14) may be written in a possibly clearer form. Consider first any Lie algebroid A on M and action of A on a smooth map $f\colon M' \to M$. Choose a connection ∇ in the vector bundle A and let $\overline{\nabla}$ denote the pullback connection in $f^!A$. Using the bracket structure in A, define the *torsion* of ∇ by

$$T_{\nabla}(X, Y) = \nabla_{aX}(Y) - \nabla_{aY}(X) - [X, Y];$$

this defines an alternating form $T_{\nabla}\colon A \oplus A \to A$. Denote the pullback of T_{∇} to $f^!A \oplus f^!A \to f^!A$ by $\overline{T}_{\nabla}$. Now the bracket in $A \lessdot f$ may be written

$$[X', Y'] = \overline{\nabla}_{a'X'}(Y') - \overline{\nabla}_{a'Y'}(X') - \overline{T}_{\nabla}(X', Y')$$

for $X', Y' \in \Gamma(f^!A)$. See [4, p.213].

Returning to the case of (14), choose a connection ∇ in A. Denote the pullback connection in $\beta^!A \to G$ by $\overline{\nabla}^{\beta}$ and the pulled–back torsion similarly. Temporarily denote the map

$\widetilde{\beta}\circ\Delta^{-1}$ by Φ. This is another action morphism of Lie algebroids, with base map β. The induced isomorphism $\Phi^!\colon A \lessdot \alpha \to A \lessdot \beta$ is $(g,X) \longmapsto (g, gX)$. The module of sections of the domain of this map is generated by the pullback sections $\overline{X}^\alpha$, where

$$\overline{X}^\alpha(g) = (g, X(\alpha g)),$$

for $X \in \Gamma A$. Denote the image of $\overline{X}^\alpha$ under $\Phi^!$ by $X^\star$; thus $X^\star(g) = (g, gX(\alpha g))$. Now equation (14) is equivalent to the condition that, for all $X, Y \in \Gamma A$,

$$[X^\star, Y^\star] = \overline{\nabla}^\beta_{X^\S}(Y^\star) - \overline{\nabla}^\beta_{Y^\S}(X^\star) - \overline{T}^\beta_\nabla(X^\star, Y^\star). \tag{15}$$

In the case of Poisson Lie groups this reduces to (9).

3 Poisson actions of groups

Consider a Poisson action $\sigma\colon G\times P \to P$, $(g,u)\longmapsto gu$, of a Poisson group G on a Poisson manifold P. Associated with σ is the infinitesimal action $\mathfrak{g}\to\mathcal{X}(P)$, $X\longmapsto X^\dagger$, given by

$$X^\dagger(u) = T_1(g\longmapsto gu)(X)$$

for $u\in P$. Dualizing the infinitesimal action gives a map $\mathfrak{p}\colon T^*P\to\mathfrak{g}^*$ which is a morphism of Lie algebroids by a result of Xu [30, 6.1]. This map plays a key role in what follows, and needs a name; we call it the *pith*. Since $(\mathrm{Ad}\, gX)^\dagger = (\sigma_g)_*(X^\dagger)$, where $\sigma_g\colon P\to P$ is the partial map, it follows that $\mathfrak{p}(g\varphi) = \mathrm{Ad}^*_g(\mathfrak{p}(\varphi))$.

We lift the action of G on P to an action of $T^*G \Longrightarrow \mathfrak{g}^*$ on $\mathfrak{p}\colon T^*P\to\mathfrak{g}^*$ by

$$\theta\varphi = g\varphi = \varphi\circ T(\sigma_{g^{-1}})$$

where $\theta\in T^*_gG$ and $\varphi\in T^*P$; see [21]. Form the action groupoid $\Omega = T^*G\lessdot T^*P \Longrightarrow T^*P$. Elements of Ω are pairs $(\theta,\varphi)\in T^*G\times T^*P$ such that $\theta\circ T(L_g) = \mathfrak{p}(\varphi)$, and the groupoid structure on Ω is

$$\widetilde{\alpha}(\theta,\varphi) = \varphi, \quad \widetilde{\beta}(\theta,\varphi) = \theta\varphi, \quad (\theta', g\varphi)(\theta,\varphi) = (\theta'\theta,\varphi).$$

Because both $\widetilde{\alpha}\colon T^*G\to\mathfrak{g}^*$ and $\mathfrak{p}$ are Lie algebroid morphisms, it follows [4] that the pullback Ω is a Lie subalgebroid of the product $T^*G\times T^*P$, which is a Lie algebroid over $G\times P$.

Theorem 3.1 *With the structure just defined, $(\Omega; G\lessdot P, T^*P; P)$ is a vacant $\mathcal{LA}$-groupoid.*

PROOF. We will prove the result by verifying the conditions of Theorem 2.9. The actions in this case are

$$(g,u)\varphi = g\varphi, \qquad \varphi^\S(g,u) = \pi^\#_G(\mathfrak{p}(\varphi)\circ T(L_{g^{-1}})) + \pi^\#_P(\varphi). \tag{16}$$

We use the Poisson action condition [11] in the form

$$\langle\psi, \pi^\#_P(\varphi)\rangle = \langle\psi\circ T(\sigma_g), \pi^\#_P(\varphi\circ T(\sigma_g))\rangle + \langle\psi\circ T(\sigma_u), \pi^\#_G(\varphi\circ T(\sigma_u))\rangle \tag{17}$$

where $\varphi,\psi\in T^*_{gu}P$.

To check (13) we apply $T(\beta)$ to $\varphi^{\S}(g,u)$, where $\beta\colon G\times P\to P$ is the action itself. Writing $\bullet$ for the tangent action $TG\times TP\to TP$, we get

$$\pi_G^{\#}(\mathfrak{p}(\varphi)\circ T(L_{g^{-1}}))\bullet\pi_P^{\#}(\varphi)=T(\sigma_u)(\pi_G^{\#}(\varphi\circ T(\sigma_u)\circ T(L_{g^{-1}})))+T(\sigma_g)(\pi_P^{\#}(\varphi)).$$

Pairing this with any $\psi\in T^*_{gu}P$ and using (17), we get

$$\begin{aligned}\langle\psi\circ T(\sigma_u),\pi_G^{\#}(\varphi\circ T(\sigma_{g^{-1}})\circ T(\sigma_u))\rangle+\langle\psi\circ T(\sigma_g),\pi_P^{\#}(\varphi)\rangle&\\ =\langle\psi,\pi_P^{\#}(\varphi\circ T(\sigma_{g^{-1}}))\rangle=\langle\psi,\pi_P^{\#}(g\varphi)\rangle&\end{aligned}$$

and $\pi_P^{\#}(g\varphi)$ is indeed the RHS of (13) in this case.

For condition (12), note first that the tangent groupoid for $G\lessdot P\Longrightarrow P$ is the action groupoid $TG\lessdot TP\Longrightarrow TP$ for the tangent action $\bullet$ of the group TG on TP. So we must prove that

$$\begin{aligned}&\pi_G^{\#}(\mathfrak{p}(\varphi)\circ T(L_{g^{-1}h^{-1}}))+\pi_P^{\#}(\varphi)=\\ &\qquad\Big[(\pi_G^{\#}(\mathfrak{p}(g\varphi)\circ T(L_{h^{-1}}))+\pi_P^{\#}(g\varphi))\Big]\bullet\Big[(\pi_G^{\#}(\mathfrak{p}(\varphi)\circ T(L_{g^{-1}}))+\pi_P^{\#}(\varphi))\Big].\end{aligned}$$

Computing in $TG\lessdot TP$, the RHS is

$$\Big[\pi_G^{\#}(\mathfrak{p}(g\varphi)\circ T(L_{h^{-1}}))\bullet\pi_G^{\#}(\mathfrak{p}(\varphi)\circ T(L_{g^{-1}}))\Big]+\pi_P^{\#}(\varphi)$$

and equality for the TP components is clear. For the TG components we need only recall that $\pi_G^{\#}\colon T^*G\to TG$ is a groupoid morphism, and that

$$(\mathfrak{p}(g\varphi)\circ T(L_{h^{-1}}))(\mathfrak{p}(\varphi)\circ T(L_{g^{-1}}))=\mathfrak{p}(\varphi)\circ T(L_{g^{-1}h^{-1}})$$

in $T^*G\Longrightarrow\mathfrak{g}^*$.

For condition (14), consider $\varphi\in\Lambda^1(P)$ and $\theta\oplus\psi\in\Gamma(T^*G\times_{\mathfrak{g}^*}T^*P)$. Thus $\theta(g,u)\circ T(L_g)=\psi(g,u)\circ T(\sigma_u)$ for all $g\in G,u\in P$. The condition that $\theta\oplus\psi=\widetilde{\varphi}$ is that $\theta(g,u)\bullet\psi(g,u)=\varphi(gu)$ for all $g\in G,u\in P$, and this expands to

$$\langle\theta(g,u),X\rangle+\langle\psi(g,u),V\rangle=\langle\varphi(gu),X\bullet V\rangle\tag{18}$$

for all $X\in T_gG,V\in T_uP$, where $X\bullet V=T(\sigma)(X\oplus V)$. Equivalently,

$$\theta(g,u)=\varphi(gu)\circ T(\sigma_u),\qquad\psi(g,u)=\varphi(gu)\circ T(\sigma_g)\qquad\text{for all } g\in G,\ u\in P.$$

Define $E\colon G\times P\to G\times P$ by $(g,u)\longmapsto(g,gu)$. This is a Poisson diffeomorphism, and the induced map $E^*\colon\Lambda^1(G\times P)\to\Lambda^1(G\times P)$ is given by

$$\langle E^*(\theta\oplus\psi)(g,u),X\oplus V\rangle=\langle\theta(g,gu),X\rangle+\langle\psi(g,gu),X\bullet V\rangle.$$

So we have proved the following lemma.

Lemma 3.2 *With the above notation,* $\theta\oplus\psi=\widetilde{\varphi}$ *if and only if* $E^*(0\oplus\varphi)=\theta\oplus\psi$.

Now $0 \oplus \varphi \in \Gamma(T^*G \times_{\mathfrak{g}^*} T^*P)$ depends on the P variable only, so $[0 \oplus \varphi_1, 0 \oplus \varphi_2] = 0 \oplus [\varphi_1, \varphi_2]$. Since E is a Poisson diffeomorphism, E^* preserves the Lie algebroid brackets, and so it follows that if $\theta_1 \oplus \psi_1 = \widetilde{\varphi_1}$, $\theta_2 \oplus \psi_2 = \widetilde{\varphi_2}$, then $[\theta_1 \oplus \psi_1, \theta_2 \oplus \psi_2] = \widetilde{[\varphi_1, \varphi_2]}$, and so $[\widetilde{\varphi_1}, \widetilde{\varphi_2}] = \widetilde{[\varphi_1, \varphi_2]}$ for all $\varphi_1, \varphi_2 \in \Lambda^1(P)$.

This completes the proof of Theorem 3.1. ∎

We now need to show how this vacant $\mathcal{LA}$-groupoid structure induces the matched pair constructed in [9]. We first briefly summarize the differentiation process for a general $\mathcal{LA}$-groupoid. See [16, §2] for more detail.

Consider any $\mathcal{LA}$-groupoid as in Figure 1(a). Applying the Lie functor to the groupoid structures gives a double vector bundle as in Figure 2(a)

$$\begin{array}{ccc} A\Omega & \xrightarrow{\;\overset{\vee}{q}\;} & A \\ {\scriptstyle A(\widetilde{q})}\downarrow & & \downarrow{\scriptstyle q} \\ AH & \xrightarrow[\;q_H\;]{} & M. \end{array} \qquad\qquad \begin{array}{ccc} \mathfrak{g} \times \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ \mathfrak{g} & \longrightarrow & \{\cdot\} \end{array}$$

(a) (b)

Figure 2:

in which $A\Omega \to AH$ has a Lie algebroid structure obtained by prolongation of the Lie algebroid structure on Ω [16, §2]; with respect to these structures, $A\Omega$ is a double Lie algebroid [15]. The map $\Delta\colon \Omega \to H \times_M A$ differentiates to the corresponding map $A\Omega \to AH \oplus A$; we call this the *double projection.* It follows that if Ω is vacant, then the double projection is a diffeomorphism, that is, $A\Omega$ is isomorphic as a double vector bundle to $AH \times_M A$. In [17] this is called a vacant double Lie algebroid and it is shown that a double Lie algebroid which is vacant defines, and is equivalent to, a matched pair structure on its two side Lie algebroids.

Here we only cite the formulas by which the representations in a vacant $\mathcal{LA}$-groupoid give rise to the matched pair of Lie algebroids. See [16, 4.5] for the details. The following definition is from Mokri [22, 4.2].

Definition 3.3 *Let A and B be Lie algebroids on base M, with anchors a and b, and let $\rho\colon A \to \mathrm{CDO}(B)$ and $\chi\colon B \to \mathrm{CDO}(A)$ be representations of A on the vector bundle B and of B on the vector bundle A. Then A and B together with ρ and χ form a* matched pair *if the following equations hold for all $X, X_1, X_2 \in \Gamma A$, $Y, Y_1, Y_2 \in \Gamma B$:*

$$\begin{aligned} \rho_X([Y_1, Y_2]) &= [\rho_X(Y_1), Y_2] + [Y_1, \rho_X(Y_2)] + \rho_{\chi_{Y_2}(X)}(Y_1) - \rho_{\chi_{Y_1}(X)}(Y_2), \\ \chi_Y([X_1, X_2]) &= [\chi_Y(X_1), X_2] + [X_1, \chi_Y(X_2)] + \chi_{\rho_{X_2}(Y)}(X_1) - \chi_{\rho_{X_1}(Y)}(X_2), \\ a(\chi_Y(X)) - b(\rho_X(Y)) &= [b(Y), a(X)]. \end{aligned}$$

Here $\mathrm{CDO}(E)$, for any vector bundle E, is the vector bundle the sections of which are those first or zeroth order differential operators $D\colon \Gamma E \to \Gamma E$ for which there is a vector field X on M with $D(f\mu) = fD(\mu) + X(f)\mu$ for all $f \in C^\infty(M)$, $\mu \in \Gamma E$. With anchor $D \longmapsto X$ and the usual bracket, $\mathrm{CDO}(E)$ is a Lie algebroid (see [12, III§2]).

Now consider a vacant $\mathcal{LA}$-groupoid as in Figure 2(a). There is an action of H on A, denoted $(h, Y) \longmapsto hY$, as in (11), and an action of A on $\alpha\colon H \to M$, as in (10), denoted $Y \longmapsto Y^\S$. Applying the complete prolongation process of [19] to the latter, we obtain an action $Y \longmapsto \widetilde{Y^\S}$ of A on AH, for which the corresponding representation $\chi\colon A \to \mathrm{CDO}(AH)$ is

$$\chi_Y(X) = [Y^\S, \vec{X}] \circ 1; \tag{19}$$

see [19, 3.14]. The action of H on A induces a representation $\rho\colon AG \to \mathrm{CDO}(A)$ by a standard procedure [12, III 4.6], giving

$$\rho_X(Y)(m) = -\left.\frac{d}{dt}\mathrm{Exp}\, tX(f_{-t}(m))Y(f_{-t}(m))\right|_0 \tag{20}$$

where $X \in \Gamma AH, Y \in \Gamma A$, and f_t is a local flow of $a(X)$.

It now follows, as an application of the abstract approach in [17, §6], that this ρ and this χ satisfy the matched pair equations of 3.3. Theorem 3.4 summarizes this.

Theorem 3.4 *Given a vacant $\mathcal{LA}$-groupoid, the representations* (19), (20) *make (AH, A) a matched pair of Lie algebroids.*

Theorem 3.5 *Let $G \times P \to P$ be a Poisson action. Then the representations* (19), (20) *induced by the vacant $\mathcal{LA}$-groupoid* 3.1 *are*

$$\langle \rho_X(\varphi), V\rangle = a(X)\langle\varphi, V\rangle - \langle\varphi, [X^\dagger, V]\rangle - \langle\varphi, (V(X))^\dagger\rangle, \quad \chi_\varphi(X) = \pi_P^\#(\varphi)(X) + \mathrm{ad}^*_{\mathfrak{p}(\varphi)}(X), \tag{21}$$

where $X\colon P \to \mathfrak{g}$, $\varphi \in \Lambda^1(P)$, $V \in \mathcal{X}(P)$. Here $V(X)$ is the Lie derivative of the vector valued map X.

PROOF. In each case it suffices to take X constant; the result for general X then follows by linearity over functions or Leibniz rules. When X is constant, the first formula is just the complete cotangent lift of the action of $\mathfrak{g}$ on P (as in [19]), and the action of G on T^*P is just the contragredient of the tangent lift.

For the second, we must prove that

$$[\vec{X}, \varphi^\dagger] \circ 1 = -\mathrm{ad}^*_{\mathfrak{p}(\varphi)}(X)$$

for X constant. For $X \in \mathfrak{g}$, the right–invariant vector field $\vec{X}$ on $G \lessdot P$ has flow $f_t(g, u) = (\exp tX\, g, u)$. Considering $1_u = (1, u)$ we have

$$\varphi^\dagger(\exp -tX, u) = \pi_G^\#(\mathfrak{p}(\varphi(u)) \circ T(L_{\exp tX})) + \pi_P^\#(\varphi(u)).$$

Applying $T(f_t)$ and differentiating, the TP term vanishes, and we have

$$\left.\frac{d}{dt}T(L_{\exp tX})\pi_G^\#(\mathfrak{p}(\varphi(u)) \circ T(L_{\exp tX}))\right|_0,$$

which is $-\mathrm{ad}^*_{\mathfrak{p}(\varphi(u))}(X)$. The result follows. ∎

$$\begin{array}{ccc} T^*G \lessdot T^*P & \rightrightarrows & T^*P \\ \downarrow & & \downarrow \\ G \lessdot P & \rightrightarrows & P \end{array} \qquad\qquad \begin{array}{ccc} (\mathfrak{g} \bowtie \mathfrak{g}^*) \lessdot T^*P & \longrightarrow & T^*P \\ \downarrow & & \downarrow \\ \mathfrak{g} \lessdot P & \longrightarrow & P. \end{array}$$

(a) (b)

Figure 3:

Apart from the sign conventions, (21) are precisely the representations of [9, §4]. The corresponding vacant double Lie algebroid has the form shown in Figure 3(b). Here the horizontal Lie algebroid is the action structure $(\mathfrak{g} \lessdot \mathfrak{g}^*) \lessdot T^*P$ from the action of the Lie algebroid $\mathfrak{g} \lessdot \mathfrak{g}^*$ on the pith $\mathfrak{p}\colon T^*P \to \mathfrak{g}^*$. The vertical structure is the pullback Lie algebroid $(\mathfrak{g} \gtrdot \mathfrak{g}^*) \times_{\mathfrak{g}^*} T^*P$ of the action Lie algebroid $\mathfrak{g} \gtrdot \mathfrak{g}^*$, the base of which is $\mathfrak{g}$, and T^*P, over the action morphism $\mathfrak{g} \gtrdot \mathfrak{g}^* \to \mathfrak{g}^*$ and the pith. The double projection of Figure 3(b) is

$$(\mathfrak{g} \bowtie \mathfrak{g}^*) \lessdot T^*P \to (\mathfrak{g} \lessdot P) \oplus T^*P.$$

This map, which we denote Υ, is an isomorphism in three distinct senses. It is an isomorphism of Lie algebroids over $\mathfrak{g} \lessdot P$ from the pullback Lie algebroid $(\mathfrak{g} \gtrdot \mathfrak{g}^*) \times_{\mathfrak{g}^*} T^*P$ to the action Lie algebroid $T^*P \lessdot (\mathfrak{g} \times P)$ corresponding to χ in (21). Secondly, it is an isomorphism of Lie algebroids over T^*P from the action Lie algebroid $(\mathfrak{g} \lessdot \mathfrak{g}^*) \lessdot T^*P$ to the action Lie algebroid $(\mathfrak{g} \lessdot P) \lessdot T^*P$ corresponding to ρ in (21). Lastly, it is an isomorphism over P of the bicrossproduct Lie algebroids

$$(\mathfrak{g} \bowtie \mathfrak{g}^*) \bowtie T^*P \to (\mathfrak{g} \lessdot P) \bowtie T^*P, \tag{22}$$

where the RHS is the form in which Lu [9] expressed the matched pair.

It follows immediately that the canonical projection $\Phi\colon (\mathfrak{g} \bowtie \mathfrak{g}^*) \lessdot T^*P \to \mathfrak{g} \bowtie \mathfrak{g}^*$ is a morphism of double Lie algebroids; that is, it is a morphism of Lie algebroids over $\mathfrak{p}\colon T^*P \to \mathfrak{g}^*$, and a morphism of Lie algebroids over the canonical projection $\mathfrak{g} \lessdot P \to \mathfrak{g}$. A morphism of vacant double Lie algebroids induces a morphism of the corresponding matched pairs, and hence of their bicrossproducts

$$(\mathfrak{g} \bowtie \mathfrak{g}^*) \bowtie T^*P \to \mathfrak{g} \bowtie \mathfrak{g}^*;$$

this gives [9, 4.4]. Note that Φ, as a morphism over $\mathfrak{p}$, is always an action morphism, but this is not generally so for Φ over $\mathfrak{g} \lessdot P \to \mathfrak{g}$, or as a map of bicrossproducts; see [9, §5].

The vacant $\mathcal{LA}$-groupoid underlying (22) has implications for the analysis of Poisson homogeneous spaces given in [9]; these will be studied elsewhere.

Lastly in this section, note that the actions (21) require only the infinitesimal data from $G \times P \to P$. There is thus the following purely infinitesimal result. The proof follows by combining [9, 4.1] with the results of [17, §6].

Theorem 3.6 *Let* $\mathfrak{g} \to \mathcal{X}(P)$ *be a Poisson action of a Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ *on a Poisson manifold* P*, with pith* $\mathfrak{p}$*. Then the actions* (21) *define the vacant double Lie algebroid in Figure* 3(b)*.*

4 Reduction for Poisson group actions

Consider a Poisson action $\sigma\colon G \times P \to P$ of a Poisson group G on a Poisson manifold P. There is a natural map of double structures from the $\mathcal{LA}$-groupoid $T^*G \lessdot T^*P$ in Figure 3(a) to the $\mathcal{LA}$-groupoid T^*G in Figure 1(b). Namely, the map $T^*P \to \mathfrak{g}^*$ is the pith and the maps $T^*G \lessdot T^*P \to T^*G$ and $G \lessdot P \to P$ are the projections onto the first factors; denote them by Ψ and ψ. The two projections are action morphisms of groupoids, for the action of T^*G on the pith and the action of G on P, respectively. Further, Ψ is a morphism of Lie algebroids over ψ.

Denote by K the kernel of $\mathfrak{p}$; we assume, for simplicity in the brief treatment in this section, that $\mathfrak{p}$ has constant rank. Similarly denote by $\widetilde{K}$ the kernel of Ψ as a morphism over ψ. Both K and $\widetilde{K}$ are Lie subalgebroids. It is immediate that, as a vector bundle, $\widetilde{K}$ is $G \times K$. It also follows, from diagrammatics or by a direct check, that the groupoid structure of $T^*G \lessdot T^*P \Longrightarrow T^*P$ restricts to $\widetilde{K} \Longrightarrow K$, and this groupoid structure is an action groupoid, $G \lessdot K \Longrightarrow K$, for the action of G on K which is the restriction of the cotangent lift. These two kernels thus form another $\mathcal{LA}$-groupoid, similar to Figure 3(a).

Any groupoid defines an orbit equivalence relation on its base. Suppose, again for simplicity here, that the orbit relations defined by the groupoids $G \lessdot K \Longrightarrow K$ and $G \lessdot P \Longrightarrow P$ correspond to quotient manifolds K/G and P/G. Then, by the next proposition, the vertical Lie algebroid structures in Figure 3(a) descend to give K/G a Lie algebroid structure over P/G.

Proposition 4.1 *Consider an* $\mathcal{LA}$*-groupoid* $(\Omega; H, A; M)$ *as in* Figure 1(a) *(but not necessarily vacant). Assume that the orbit equivalence relations defined by* $\Omega \Longrightarrow A$ *and* $H \Longrightarrow M$ *correspond to quotient manifolds* A' *and* M'*. Then there is a unique structure of Lie algebroid on* A' *over base* M' *such that the natural maps* $A \to A'$ *and* $M \to M'$ *form a morphism of Lie algebroids.*

This is an immediate consequence of the treatment of Lie algebroid quotients in [4, §4]. There is a corresponding result for double groupoids in [13, §3]. The proof of the next result will be given in a subsequent article.

Theorem 4.2 *Applying* Proposition 4.1 *to the* $\mathcal{LA}$*-groupoid in* Figure 3(a)*, the orbit Lie algebroid is the cotangent Lie algebroid of* P/G*.*

Before we turn to examples, we sketch briefly the corresponding approach on the level of double groupoids. Suppose that Σ is a symplectic double groupoid for G in the sense of [10]. Assume that $\Pi \Longrightarrow P$ is an α–simply connected symplectic groupoid for P. Then we may construct double groupoids which are directly analogous to, and global forms of, the $\mathcal{LA}$-groupoids in Figures 3(a) and 1(b), take the kernels of the corresponding morphisms, take the quotient manifolds defined by the horizontal groupoid structures, and thereby obtain a symplectic groupoid for the reduced Poisson space. The details will be given in a subsequent article.

Example 4.3 Suppose that $G \times P \to P$ is a Hamiltonian action of a connected Lie group G on a connected symplectic manifold P, with equivariant moment map $\mu\colon P \to \mathfrak{g}^*$. Assume, for simplicity here, that $Q = P/G$ is a smooth manifold with the projection $P \to Q$ complete as a Poisson map, and that μ has constant rank, is complete, and has connected fibres. (In the terminology of [28, §5], the action is *completely regular.*)

The kernel K of the pith $\mathfrak{p}\colon T^*P \to \mathfrak{g}^*$ is $(\operatorname{im} a)^\circ$, the annihilator of the image of the infinitesimal action $a\colon P \times \mathfrak{g} \to TP$. Because P is symplectic, we can identify the Lie algebroid T^*P with TP, and K then corresponds to $(\operatorname{im} a)^\perp$. If P is simply–connected, then $\mathfrak{p}$ integrates to a G–equivariant morphism $\mathfrak{P}\colon P \times P \to \mathfrak{g}^*$ and $(\operatorname{im} a)^\perp$ is an involutive and G–invariant distribution on P. Now the usual conditions on μ ensure that $\mathfrak{P}(v, u) = \mu(v) - \mu(u)$ and $(\operatorname{im} a)^\perp = T^\mu P$. The Lie groupoid corresponding to $T^\mu P$ is $R(\mu) = \{(v, u) \in P \times P \mid \mu(v) = \mu(u)\} = \mathfrak{P}^{-1}(0)$. Finally, the quotient Lie groupoid $R(\mu)/G \rightrightarrows Q$ has a natural symplectic structure. In terms of [28], the symplectic groupoid $R(\mu)/G$ arises from the symplectic affinoid structure.

The following example emerged from discussions with J.–H. Lu.

Example 4.4 Suppose that P is itself a Poisson Lie group, with G a closed Poisson subgroup and $G \times P \to P$ the group multiplication. The Lie algebroid T^*P is isomorphic to the action Lie algebroid $\mathfrak{b}^* \lessdot P$ where we write $\mathfrak{b}$ for the Lie algebra of P. In these terms the pith $P \times \mathfrak{b}^* \to \mathfrak{g}^*$ is fibrewise the restriction and so the kernel K is $P \times \mathfrak{g}^\perp$. It follows now from 4.2 that the cotangent Lie algebroid of P/G is $(P \times \mathfrak{g}^\perp)/G$, the quotient over the canonical action of G of the action Lie algebroid $\mathfrak{g}^\perp \lessdot P$.

In this case the pith always integrates, giving an equivariant morphism of Lie groupoids $\Pi \to G^*$ where Π is the α–simply connected symplectic groupoid for P and G^* a dual group for G. There is rarely a moment map.

5 Concluding remarks

The approach given in §3—§4 extends to general actions of Poisson groupoids and, suitably modified, to actions of abstract Lie bialgebroids. In particular §4 applies to actions of symplectic groupoids, in which case it is the method given by Xu [29]. The method also extends to Poisson reduction in the sense of Marsden and Raţiu [20]. All these developments will follow in another article.

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References

[1] A. Coste, P. Dazord, and A. Weinstein. Groupoïdes symplectiques. In *Publications du Département de Mathématiques de l'Université de Lyon, I*, number 2/A-1987, pages 1–65, 1987.

[2] A. Cannas da Silva and A. Weinstein. *Geometric models for noncommutative algebras*, volume 10 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, 1999.

[3] V. G. Drinfel'd. On Poisson homogeneous spaces of Poisson–Lie groups. *Theo. Math. Phys.*, 95(2):226–227, 1993.

[4] P. J. Higgins and K. C. H. Mackenzie. Algebraic constructions in the category of Lie algebroids. *J. Algebra*, 129:194–230, 1990.

[5] J. Huebschmann. Poisson cohomology and quantization. *J. Reine Angew. Math.*, 408:57–113, 1990.

[6] M. V. Karasev. The Maslov quantization conditions in higher cohomology and analogs of notions developed in Lie theory for canonical fibre bundles of symplectic manifolds. I, II. *Selecta Math. Soviet.*, 8:213–234, 235–258, 1989. Preprint, Moscov. Inst. Electron Mashinostroeniya, 1981, deposited at VINITI, 1982.

[7] M. V. Karasev. Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50:508–538, 638, 1986. English translation: Math. USSR–Izv. 28 (1987), no. 3, 497–527.

[8] Jiang-Hua Lu. Momentum mappings and reduction of Poisson actions. In *Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)*, pages 209–226. Springer, New York, 1991.

[9] Jiang-Hua Lu. Poisson homogeneous spaces and Lie algebroids associated to Poisson actions. *Duke Math. J.*, 86:261–304, 1997.

[10] Jiang-Hua Lu and A. Weinstein. Groupoïdes symplectiques doubles des groupes de Lie-Poisson. *C. R. Acad. Sci. Paris Sér. I Math.*, 309:951–954, 1989.

[11] Jiang-Hua Lu and A. Weinstein. Poisson Lie groups, dressing transformations, and Bruhat decompositions. *J. Differential Geom.*, 31:501–526, 1990.

[12] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry.* London Mathematical Society Lecture Note Series, no. 124. Cambridge University Press, 1987.

[13] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry, I. *Adv. Math.*, 94(2):180–239, 1992.

[14] K. C. H. Mackenzie. Lie algebroids and Lie pseudoalgebras. *Bull. London Math. Soc.*, 27(2):97–147, 1995.

[15] K. C. H. Mackenzie. Drinfel'd doubles and Ehresmann doubles for Lie algebroids and Lie bialgebroids. *Electron. Res. Announc. Amer. Math. Soc.*, 4:74–87, 1998.

[16] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry, II. *Adv. Math. (to appear)*, 2000. 27pp.

[17] K. C. H. Mackenzie. Notions of double for Lie algebroids. Submitted, 2000. 30 pages.

[18] K. C. H. Mackenzie and Ping Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.*, 73(2):415–452, 1994.

[19] K. C. H. Mackenzie and Ping Xu. Classical lifting processes and multiplicative vector fields. *Quarterly J. Math. Oxford (2)*, 49:59–85, 1998.

[20] Jerrold E. Marsden and Tudor Raţiu. Reduction of Poisson manifolds. *Lett. Math. Phys.*, 11(2):161–169, 1986.

[21] K. Mikami and A. Weinstein. Moments and reduction for symplectic groupoids. *Publ. Res. Inst. Math. Sci.*, 24:121–140, 1988.

[22] T. Mokri. Matched pairs of Lie algebroids. *Glasgow Math. J.*, 39:167–181, 1997.

[23] I. Vaisman. *Lectures on the Geometry of Poisson Manifolds*, volume 118 of *Progress in Mathematics*. Birkhäuser, 1994.

[24] A. Weinstein. The local structure of Poisson manifolds. *J. Differential Geom.*, 18:523–557, 1983. Errata and Addenda, same journal, 22:255, 1985.

[25] A. Weinstein. Symplectic groupoids and Poisson manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 16:101–104, 1987.

[26] A. Weinstein. Coisotropic calculus and Poisson groupoids. *J. Math. Soc. Japan*, 40:705–727, 1988.

[27] A. Weinstein. Some remarks on dressing transformations. *J. Fac. Sci. Univ. Tokyo. Sect. 1A, Math.*, 36:163–167, 1988.

[28] A. Weinstein. Affine Poisson structures. *Internat. J. Math.*, 1:343–360, 1990.

[29] Ping Xu. Symplectic groupoids of reduced Poisson spaces. *C. R. Acad. Sci. Paris Sér. I Math.*, 314:457–461, 1992.

[30] Ping Xu. On Poisson groupoids. *Internat. J. Math.*, 6(1):101–124, 1995.